3. Pokažite da red $\sum a_n$ konvergira te izračunajte sumu $\sum_{n=2}^{\infty} a_n$, gdje je

$a_n = 3\cdot\left(\frac{4^n}{5^{n+1}}\right)\ (\forall n \in \mathbb{N}_0)$

$a_n = 3\cdot\frac{4^n}{5^n 5} = \frac{3}{5}\cdot\left(\frac{4}{5}\right)^n \Rightarrow$, geometrijski red

$q = \frac{4}{5}$ $\quad |q| < 1$ red će da konvergira, u suprotnom on divergira.

Pošto su $\left|\frac{4}{5}\right| < 1$ naš red će da konvergira.

$$\sum_{n=2}^{\infty} \frac{3}{5}\left(\frac{4}{5}\right)^n = \sum_{n=0}^{\infty} \frac{3}{5}\left(\frac{4}{5}\right)^n - \frac{3}{5} - \frac{12}{25} = \frac{\frac{3}{5}}{1-\frac{4}{5}} - \frac{3}{5} - \frac{12}{25}$$

$$= \frac{\frac{3}{\cancel{5}}}{\frac{1}{\cancel{5}}} - \frac{3}{5} - \frac{12}{25} = \frac{75-15-12}{25} = \frac{48}{25}$$

Dokažite neposredno konvergenciju sljedećeg reda i naćite sumu:

$$\sum_{n=0}^{\infty} x^{\left[\frac{n}{2}\right]} \cdot y^{\left[\frac{n+1}{2}\right]} = 1 + y + xy + xy^2 + x^2y^2 + x^2y^3 + x^3y^3 + \ldots$$

$$= \underbrace{\left(1 + xy + x^2y^2 + x^3y^3 + \ldots\right)} + y\underbrace{\left(1 + xy + x^2y^2 + \ldots\right)}$$

Ovo su dva geometrijska reda sa $q = x\cdot y$

Ako je $|x\cdot y| < 1$ red konvergira, a ako je $|x\cdot y| > 1$ red divergira

Suma geometrijskog reda: za $|xy| < 1$:

$$\frac{1}{1-xy} + y\cdot\frac{1}{1-xy} = \frac{1+y}{1-xy}$$, $\Rightarrow$ red konvergira, suma konačna

5. Odredite prirodni domen, ispitajte ograničenost, parnost/neparnost, periodičnost i odredite osnovni period realne funkcije f jedne realne promjenljive zadane formulom

$$f(x) = 6\,tg\,\frac{mx}{10} - 7\,tg\,\frac{x}{7}, \quad m=34$$

$$f(x) = 6\,tg\,\frac{34x}{10} - 7\,tg\,\frac{x}{7}$$

$$f(x) = \frac{6\sin\frac{17x}{5}}{\cos\frac{17x}{5}} - 7\,\frac{\sin\frac{x}{7}}{\cos\frac{x}{7}}$$

$$\cos\frac{17x}{5} \neq 0 \quad \wedge \quad \cos\frac{x}{7} \neq 0$$

$$\cos\frac{17x}{5} \neq \cos\left(\frac{\pi}{2}+k\pi\right) \quad \wedge \quad \cos\frac{x}{7} \neq \cos\left(\frac{\pi}{2}+k\pi\right)$$

$$\frac{17x}{5} \neq \frac{\pi}{2}+k\pi \;\Big/ \cdot\frac{5}{17} \quad \wedge \quad \frac{x}{7} \neq \frac{\pi}{2}+k\pi \;\Big/ \cdot 7$$

$$x \neq \frac{5\pi}{34} + \frac{5k\pi}{17} \quad \wedge \quad x \neq \frac{7\pi}{2} + 7k\pi$$

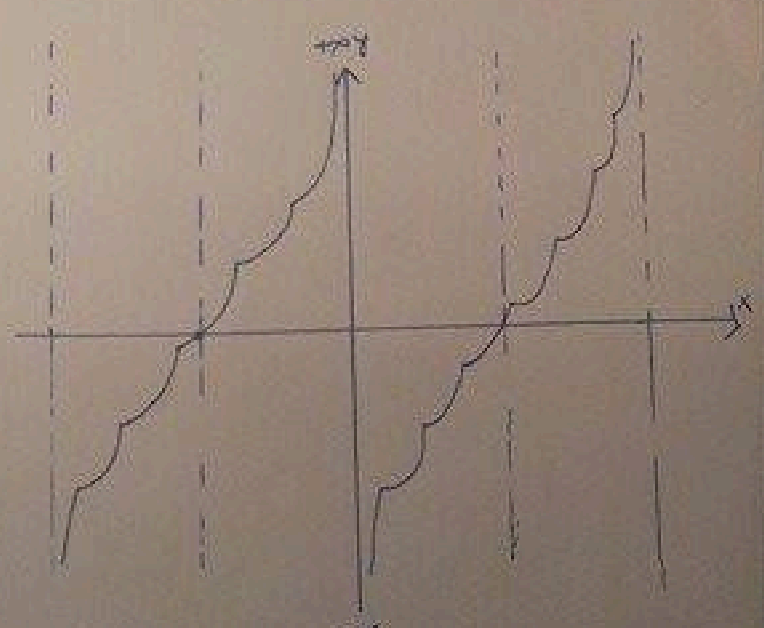
$$D(f) = \left\{ x\in\mathbb{R} \;\middle|\; x \neq \frac{5\pi}{34}+\frac{5k\pi}{17} \;\wedge\; x \neq \frac{7\pi}{2}+7k\pi \right\}, \; k\in\mathbb{Z}$$

$$\lim_{x\to\frac{5\pi}{34}^-} \left(\frac{6\sin\frac{17x}{5}}{\cos\frac{17x}{5}} - \frac{7\sin\frac{x}{7}}{\cos\frac{x}{7}} \right) = +\infty$$

$$\lim_{x\to\frac{5\pi}{34}^+} \left(\frac{6\sin\frac{17x}{5}}{\cos\frac{17x}{5}} - \frac{7\sin\frac{x}{7}}{\cos\frac{x}{7}} \right) = -\infty$$

$$\lim_{x\to\frac{7\pi}{2}^-} \left(\frac{6\sin\frac{17x}{5}}{\cos\frac{17x}{5}} - \frac{7\sin\frac{x}{7}}{\cos\frac{x}{7}} \right) = -\infty$$

$$\lim_{x\to\frac{7\pi}{2}^+} \left(\frac{6\sin\frac{17x}{5}}{\cos\frac{17x}{5}} - \frac{7\sin\frac{x}{7}}{\cos\frac{x}{5}} \right) = +\infty$$



$$z_1 = \sqrt{5} \cdot e^{i \operatorname{arctg} \frac{1}{2}}$$

$$z_1^{20} = (\sqrt{5})^{20} \cdot e^{i 20 \operatorname{arctg} \frac{1}{2}} = (5^{\frac{1}{2}})^{20} \cdot e^{i 20 \operatorname{arctg} \frac{1}{2}} = 5^{10} \cdot e^{i 20 \operatorname{arctg} \frac{1}{2}}$$

$$z_2 = 2 \cdot e^{i \frac{2\pi}{3}}$$

$$z_2^{15} = 2^{15} \cdot e^{i \frac{2 \cdot \pi \cdot 15}{3}} = 2^{15} \cdot e^{i 10\pi} = 2^{15} \cdot e^{i 2\pi}$$

$$\frac{z_2}{z_1} = \frac{2 \cdot e^{i \frac{2\pi}{3}}}{\sqrt{5} \cdot e^{i \operatorname{arctg} \frac{1}{2}}} = \frac{2}{\sqrt{5}} \cdot e^{i \left(\frac{2\pi}{3} - \operatorname{arctg} \frac{1}{2} \right)} = \frac{2\sqrt{5}}{5} \cdot e^{i \left(\frac{2\pi}{3} - \operatorname{arctg} \frac{1}{2} \right)}$$

$$(\bar{z}_1)^{20} = 5^{10} \cdot e^{i 20 \operatorname{arctg} (-\frac{1}{2})}$$

$$\frac{z_2^{15}}{(\bar{z}_1)^{20}} = \frac{2^{15} \cdot e^{i 2\pi}}{5^{10} \cdot e^{i 20 \operatorname{arctg} (-\frac{1}{2})}} = \frac{2^{15}}{5^{10}} \cdot e^{i \left(2\pi - 20 \operatorname{arctg} (-\frac{1}{2}) \right)}$$

1. Zadani su kompleksni brojevi:

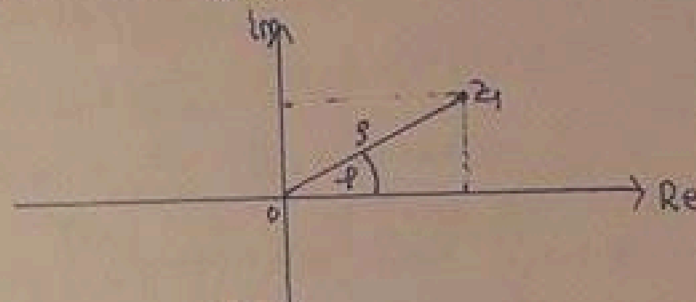
$$z_1 = 2+i \quad z_2 = -1+i\sqrt{3} \quad z_3 = \frac{1-3i}{1-i} - \frac{m+i}{2+i} \quad z_4 = 1+i\,tg\,\alpha \quad z_5 = \ln\left[i\,th\left(\frac{\pi}{2}\right)\right]$$

$\alpha \geq 0$, i - imaginarna jedinica, $m = 34$

$$z_1 = 2+i = \sqrt{5}\left(\cos\left(arctg\frac{1}{2}\right) + i\sin\left(arctg\frac{1}{2}\right)\right) = \sqrt{5}\cdot e^{i\,arctg\frac{1}{2}} = \sqrt{5}\,cis\left(arctg\frac{1}{2}\right)$$

$$\rho = \sqrt{2^2+1^2} = \sqrt{5}$$

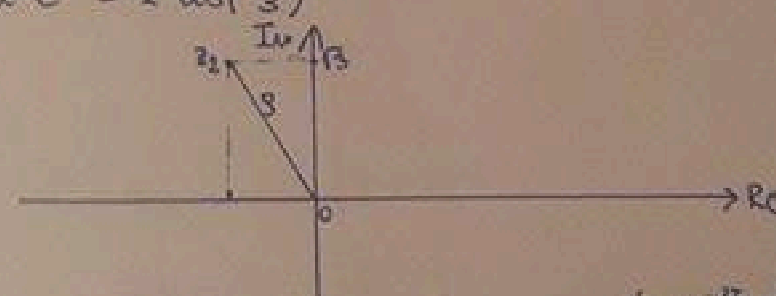
$$\varphi = arctg\frac{1}{2}$$



$$z_2 = -1+i\sqrt{3} = 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = 2\cdot e^{i\frac{2\pi}{3}} = 2\cdot cis\left(\frac{2\pi}{3}\right)$$

$$\rho = \sqrt{(-1)^2+(\sqrt{3})^2} = 2$$

$$\varphi = arctg\frac{\sqrt{3}}{-1} + \pi = -\frac{\pi}{3} + \pi = \frac{-\pi+3\pi}{3} = \frac{2\pi}{3}$$



$$z_3 = \frac{1-3i}{1-i} - \frac{34+i}{2+i} = \frac{1}{5}\sqrt{4210}\cdot\left(\cos\left(arctg\frac{-27}{59}+\pi\right) + i\sin\left(arctg\frac{-27}{59}+\pi\right)\right) = \frac{1}{5}\sqrt{4210}\cdot e^{i\left(arctg\frac{-27}{59}+\pi\right)} = \frac{1}{5}\sqrt{4210}\cdot cis\left(arctg\frac{-27}{59}+\pi\right)$$

$$z_3 = \frac{(1-3i)(2+i)-(34+i)(1-i)}{(1-i)(2+i)} = \frac{2+i-6i+3-(34-i34+i+1)}{(1-i)(2+i)} = \frac{5-5i-(35-i33)}{2+i-i2+1}$$

$$= \frac{5-5i-35+i33}{3-i} = \frac{-30+i28}{3-i}\cdot\frac{3+i}{3+i} = \frac{-90-i30+i84-28}{9+1} = \frac{-118+i54}{10} = \frac{-59}{5} + i\frac{27}{5}$$

$$\rho = \sqrt{\left(\frac{-59}{5}\right)^2 + \left(\frac{27}{5}\right)^2} = \frac{1}{5}\sqrt{4210}$$

$$\varphi = arctg\frac{\frac{27}{5}}{\frac{-59}{5}} + \pi = arctg\frac{-27}{59} + \pi$$

$$f(x) = 6\,tg\,\frac{34x}{10} - 7\,tg\,\frac{x}{7}$$

$$f(-x) = 6\,tg\left(\frac{-34x}{10}\right) - 7\,tg\left(\frac{-x}{7}\right)$$

$$f(-x) = -6\,tg\left(\frac{34x}{10}\right) + 7\,tg\left(\frac{x}{7}\right)$$

$$f(-x) = -\left(6\,tg\,\frac{34x}{10} - 7\,tg\,\frac{x}{7}\right)$$

$f(-x) = -f(x)$ Funkcija je neparna

$$T = NZS\left(\frac{5\pi}{17}, 7\pi\right) = \pi\, NZS\left(\frac{5}{17}, 7\right) = \pi\, NZS\left(\frac{5}{17}, \frac{17\cdot 7}{17}\right)$$

$$T = \pi \cdot \frac{1}{17}\, NZS(5, 119) = \frac{1}{17}\pi\, 595 = \cancel{35} \; 35\pi$$

$z_4 = 1 + i\,tg\alpha \,, \quad \alpha > 0$

$\rho = \sqrt{1^2 + tg^2\alpha} = \sqrt{1 + \frac{\sin^2\alpha}{\cos^2\alpha}} = \sqrt{\frac{\cos^2\alpha + \sin^2\alpha}{\cos^2\alpha}} = \sqrt{\frac{1}{\cos^2\alpha}} = \frac{1}{|\cos\alpha|}$

$\varphi = arctg(tg\alpha) \Rightarrow tg\varphi = tg\alpha$

$1^\circ \quad \alpha \in \left[0, \frac{\pi}{2}\right) \quad \varphi = \alpha \Rightarrow z_4 = \frac{1}{|\cos\alpha|} \cdot (\cos\alpha + i\sin\alpha) = \frac{1}{|\cos\alpha|} \cdot e^{i\alpha}$

$2^\circ \quad \alpha \in \left(\frac{\pi}{2}, \pi\right] \quad \varphi = \alpha + \pi \Rightarrow z_4 = \frac{1}{|\cos\alpha|} \cdot \left(\cos(\alpha+\pi) + i\sin(\alpha+\pi)\right) = \frac{1}{|\cos\alpha|} \cdot e^{i(\alpha+\pi)}$

$3^\circ \quad \alpha \in \left[\pi, \frac{3\pi}{2}\right) \quad \varphi = \alpha - \pi \Rightarrow z_4 = \frac{1}{|\cos\alpha|} \cdot \left(\cos(\alpha-\pi) + i\sin(\alpha-\pi)\right) = \frac{1}{|\cos\alpha|} \cdot e^{i(\alpha-\pi)}$

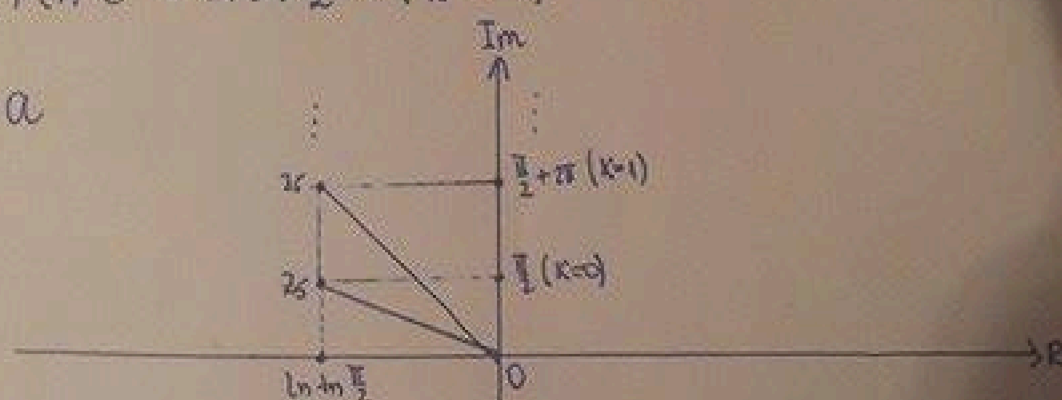
$4^\circ \quad \alpha \in \left(\frac{3\pi}{2}, 2\pi\right] \quad \varphi = \alpha \Rightarrow z_4 = \frac{1}{|\cos\alpha|} \cdot (\cos\alpha + i\sin\alpha) = \frac{1}{|\cos\alpha|} \cdot e^{i\alpha}$

$z_5 = \ln\left[i\,th\left(\frac{\pi}{2}\right)\right] = a \cdot \left(\cos\left(arctg\frac{\frac{\pi}{2}+2k\pi}{\ln th\frac{\pi}{2}} + \pi\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{\ln th\frac{\pi}{2}} + \pi\right)\right) = a \cdot e^{i\left(arctg\frac{\frac{\pi}{2}+2k\pi}{\ln th\frac{\pi}{2}} + \pi\right)}$

$z_5 = \ln th\frac{\pi}{2} + \ln i = \ln th\frac{\pi}{2} + \ln \cdot e^{i\frac{\pi}{2}} = \ln th\frac{\pi}{2} + i\left(\frac{\pi}{2} + 2k\pi\right)$

$\rho = \sqrt{\ln^2 th\frac{\pi}{2} + \left(\frac{\pi}{2} + 2k\pi\right)^2} = a$

$\varphi = arctg\,\frac{\frac{\pi}{2} + 2k\pi}{\ln th\frac{\pi}{2}} + \pi$



Glavna vrijednost argumenta kompleksnog broja z_5:

$\varphi = arctg\,\frac{\frac{\pi}{2}}{\ln th\frac{\pi}{2}} + \pi$

6. Ustanovite da li ima smisla pa izračunajte sljedeće limese:

$$\lim_{x\to-\infty}\left(x-\sqrt{x^2+px+4}\right)$$, gdje je p prirodan broj

$$x^2+px+4\geq 0$$

$$x_{12}=\frac{-p\pm\sqrt{p^2-16}}{2} \nearrow x_1=\frac{-p+\sqrt{p^2-16}}{2}$$

$$\searrow x_2=\frac{-p-\sqrt{p^2-16}}{2}$$

$$(x-x_1)(x-x_2)\geq 0$$

	$-\infty$ — x_2	x_2 — x_1	x_1 — $+\infty$
$x-x_1$	$-$	$-$ 0	$+$
$x-x_2$	$-$ 0	$+$	$+$
R	$+$	$-$	$+$

$$x\in(-\infty, x_2]\cup[x_1,+\infty)$$

Pošto je $-\infty$ tačka gomilanja onda ima smisla tražiti limes

$$\lim_{x\to-\infty}\left(x-\sqrt{x^2+px+4}\right) \quad \left|\begin{array}{l} x=-t \\ t=-x \\ x\to-\infty \\ t\to+\infty \end{array}\right|$$

$$\lim_{t\to\infty}\frac{-t+\sqrt{t^2-pt+4}}{1}\cdot\frac{-t-\sqrt{t^2-pt+4}}{-t-\sqrt{t^2-pt+4}}=\lim_{t\to\infty}\frac{\cancel{t^2}-\cancel{t^2}+pt-4 \;/:t}{-t-\sqrt{t^2-pt+4}\;/:t}$$

$$\lim_{t\to\infty}\frac{p-\cancel{\frac{4}{t}}^{0}}{-1-\sqrt{1-\cancel{\frac{p}{t}}+\cancel{\frac{4}{t^2}}^{0}}}=-\frac{p}{2}$$

2. Za niz (a_n), $a_n := \frac{n}{\sqrt[3]{n^6-1}} + \frac{n}{\sqrt[3]{n^6-2}} + \dots + \frac{n}{\sqrt[3]{n^6-n-2}}$, gdje je $n \in (\mathbb{N} \setminus \{1\})$

izračunajte a) $L_1 = \lim\limits_{n\to+\infty} (a_n)^2$ b) $L_2 = \lim\limits_{n\to+\infty} (a_n)^n$

Za rješavanje ovog zadatka koristit ćemo teorem o 2 žandara:

Neka su $(a_n), (b_n), (c_n)$ tri niza u $\mathbb{R}$, takva da je:

1° $b_n \leq a_n \leq c_n$ za svaki $n \in \mathbb{N}$ ili počevši od nekog n

2° $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = m \in \mathbb{R}$

Tada je i $\lim\limits_{n\to\infty} a_n = m$

U zadanom nizu imamo $(n+2)$ člana, a kao slični nizovi uzet ćemo:

$$b_n = \frac{n(n+2)}{\sqrt[3]{n^6-n-2}} \quad , \quad c_n = \frac{n(n+2)}{\sqrt[3]{n^6-1}}$$

Rješavamo L_1:

$$\lim_{n\to\infty}\left(\frac{n(n+2)}{\sqrt[3]{n^6-n-2}}\right)^2 = \lim_{n\to\infty}\left(\frac{(n^2+2n)/n^2}{\sqrt[3]{n^6-n-2}/n^2}\right)^2 = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{\sqrt[3]{1-\frac{1}{n^5}-\frac{2}{n^6}}}\right)^2 = \lim_{n\to\infty}\frac{1+\cancelto{0}{\frac{4}{n}}+\cancelto{0}{\frac{4}{n^2}}}{\left(\sqrt[3]{1-\cancelto{0}{\frac{1}{n^5}}-\cancelto{0}{\frac{2}{n^6}}}\right)^2} = 1$$

$$\lim_{n\to\infty}\left(\frac{n(n+2)}{\sqrt[3]{n^6-1}}\right)^2 = \lim_{n\to\infty}\left(\frac{n^2+2n}{\sqrt[3]{n^6-1}}\right)^2 = \lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{\sqrt[3]{1-\frac{1}{n^6}}}\right)^2 = \lim_{n\to\infty}\frac{1+\cancelto{0}{\frac{4}{n}}+\cancelto{0}{\frac{4}{n^2}}}{\left(\sqrt[3]{1-\cancelto{0}{\frac{1}{n^6}}}\right)^2} = 1$$

Pošto smo dobili da je $\lim\limits_{n\to\infty}(b_n)^2 = \lim\limits_{n\to\infty}(c_n)^2 = 1$, onda je i prema teoremi $\lim\limits_{n\to\infty}(a_n)^2 = 1$

Za određivanje L_2 koristit ćemo isti teorem i isti postupak:

$$\lim_{n\to\infty}\left(\frac{n(n+2)}{\sqrt[3]{n^6-n-2}}\right)^n \sim \lim_{n\to\infty}\left(\frac{n(n+2)}{\sqrt[3]{n^6}}\right)^n \Rightarrow$$ Nizovi su ekvikonvergentni jer n^6 mnogo brže raste od $-n-2$, pa to neće utjecati na rezultat i može se zanemariti

$$\lim_{n\to\infty}\left(\frac{n^2+2n}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}\cdot 2} = e^2$$

$$\lim_{n\to\infty}\left(\frac{n(n+2)}{\sqrt[3]{n^6-1}}\right)^n \sim \lim_{n\to\infty}\left(\frac{n(n+2)}{\sqrt[3]{n^6}}\right)^n \Rightarrow$$ Također nizovi su ekvikonvergentni

$$\lim_{n\to\infty}\left(\frac{n^2+2n}{n^2}\right)^n = \lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{\frac{n}{2}}\right)^{\frac{n}{2}\cdot 2} = e^2$$

Pošto smo i ovdje dobili da je $\lim\limits_{n\to\infty}(b_n)^n = \lim\limits_{n\to\infty}(c_n)^n = e^2$, onda je i $\lim\limits_{n\to\infty}(a_n)^n = e^2$.