263. Dokazati da postoji jednoznačna funkcija $y = y(x)$ zadana sljedećim jednadžbama i zatim odrediti y'_x :

a) $y^3 + 3y = x$

b) $y + \ln y = x$

c) $e^y + y = x$

Rješenje :

a) $y^3 + 3y = x \quad \left(x = x(y) \right)$

1° $y \uparrow$, $y^3 + 3y \uparrow \Rightarrow x \uparrow \Rightarrow x = x(y)$ strogo monotono rastuća fc.

2° $x = x(y)$ def. $\forall y \in \mathbb{R}$

Iz 1° i 2° $\Rightarrow \exists$ inverzna fc.

* Prema th. o izvodu inverzne fc :

$$y'_x = \frac{1}{x'_y} = \frac{1}{3y^2 + 3} = \frac{1}{3(y^2+1)} \; , \; \forall y$$

II način

$$y^3 + 3y = x \quad \Big/ '$$

$$3y^2 \cdot y' + 3 \cdot y' = 1 \Rightarrow \boxed{y' = \frac{1}{3(y^2+1)}}$$

b) $y + \ln y = x$

I način

$$y'_x = \frac{1}{x'_y} = \frac{1}{1 + \frac{1}{y}} = \frac{y}{y+1}, \quad y \neq -1$$

II način

$$y + \ln y = x \;\Big|'$$

$$y' + \frac{1}{y} \cdot y' = 1 \; ; \; y' = \frac{1}{1 + \frac{1}{y}} = \frac{y}{y+1}, \; y \neq -1$$

c) $e^y + y = x$

I način

$$y'_x = \frac{1}{x'_y} = \frac{1}{e^y + 1}, \; \forall y$$

II način

$$e^y + y = x \;\Big|'$$

$$e^y \cdot y' + y' = 1 \Rightarrow y' = \frac{1}{1 + e^y}, \; \forall y$$

2.66. Odrediti izvode sljedećih funkcija:

a) $y = x^x$ d) $y = \sqrt[x]{\frac{1}{x}}$

b) $y = x^{x^x}$ e) $y = (\sin x)^{\ln x}$

c) $y = (x^x)^x$ f) $y = \sin\left(x^{\ln x}\right)$

Rješenje:

a) $y = x^x$ odnosno

$$\ln y = x \cdot \ln x \;\Big|'$$

$$\frac{1}{y}\cdot y' = \ln x + \cancel{x}\cdot\frac{1}{\cancel{x}} \Rightarrow y' = y\,(1+\ln x) = x^x\cdot(1+\ln$$

b)

$$y = x^{x^x}$$

$$\ln y = x^x \cdot \ln x \;\Big|'$$

$$\frac{1}{y}\cdot y' = x^x\cdot(1+\ln x)\cdot\ln x + x^x\cdot\frac{1}{x}$$

$$y' = x^{x^x}\cdot\left\{x^x\left[(1+\ln x)\cdot\ln x + \frac{1}{x}\right]\right\}$$

$$y' = x^{x^x+x}\cdot\left[(1+\ln x)\cdot\ln x + \frac{1}{x}\right]$$

c) $y = x^{x\cdot x} = x^{x^2}$

$$\boxed{y' = x^{x^2+1}(2\ln x + 1)}$$

$$\ln y = x^2\cdot\ln x \;\Big|' \Rightarrow \frac{1}{y}\cdot y' = 2x\cdot\ln x + x^{\cancel{2}}\cdot\frac{1}{\cancel{x}}$$

d) $y = \frac{1}{x}^{\frac{1}{x}}$

$\ln y = \frac{1}{x} \cdot \ln \frac{1}{x} \Big|'$

$\frac{1}{y} \cdot y' = -\frac{1}{x^2} \cdot \ln \frac{1}{x} + \frac{1}{\cancel{x}} \cdot \cancel{x} \cdot \frac{-1}{x^2} \Rightarrow$

$y' = y \cdot \frac{-1}{x^2} \cdot \left(\ln \frac{1}{x} + 1 \right) \Rightarrow$

$y' = -\left(\frac{1}{x}\right)^{\frac{1}{x}} \cdot \left(\frac{1}{x}\right)^2 \left(1 + \ln \frac{1}{x} \right)$

$y' = -\left(\frac{1}{x}\right)^{\frac{1}{x}+2} \cdot \left(1 + \ln \frac{1}{x} \right), \; x \neq 0 \Big\} \cap$

$\frac{1}{x} > 0 \Big\} \; \boxed{x > 0}$

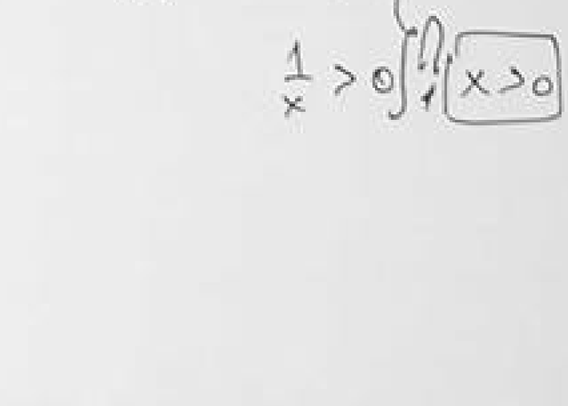
e) $y = (\sin x)^{\ln x}$

$\ln y = \ln x \cdot \ln(\sin x)$

$\frac{1}{y} \cdot y' = \frac{1}{x} \cdot \ln(\sin x) + \ln x \cdot \frac{1}{\sin x} \cdot \cos x$

$y' = (\sin x)^{\ln x} \left[\frac{1}{x} \cdot \ln(\sin x) + \ln x \cdot \operatorname{ctg} x \right]$

$\cap \begin{cases} x > 0 \\ \sin x > 0 \\ x \neq k\pi, \; k \in \mathbb{Z} \end{cases}$



$\boxed{x \in \left(k\pi, (k+1)\pi \right), \; k \in \mathbb{N}_0}$

2.56. Odrediti izvode sljedećih funkcija:

a) $y = x^x$

b) $y = x^{x^x}$

c) $y = (x^x)^x$

d) $y = \sqrt[x]{\frac{1}{x}}$

e) $y = (\sin x)^{\ln x}$

f) $y = \sin(x^{\ln x})$

Rješenje:

a) $y = x^x$

odnosno

$\ln y = x \cdot \ln x \quad \Big|'$

$\frac{1}{y} \cdot y' = \ln x + \cancel{x} \cdot \frac{1}{\cancel{x}} \Rightarrow y' = y(1 + \ln x) = x^x \cdot (1 + \ln$

b) $y = x^{x^x}$

$\ln y = x^x \cdot \ln x \quad \Big|'$

$\frac{1}{y} \cdot y' = x^x \cdot (1 + \ln x) \cdot \ln x + x^x \cdot \frac{1}{x}$

$y' = x^{x^x} \cdot \left\{ x^x \left[(1 + \ln x) \cdot \ln x + \frac{1}{x} \right] \right\}$

$y' = x^{x^x + x} \cdot \left[(1 + \ln x) \cdot \ln x + \frac{1}{x} \right]$

c) $y = x^{x \cdot x} = x^{x^2}$

$\ln y = x^2 \cdot \ln x \quad \Big|' \Rightarrow \frac{1}{y} \cdot y' = 2x \cdot \ln x + x^{\cancel{2}} \cdot \frac{1}{\cancel{x}}$

$\boxed{y' = x^{x^2+1}(2 \ln x + 1)}$

270. Za koje vrijednosti parametra α funkcija:

$$f(x) = \begin{cases} x^{\alpha} \cdot \sin\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

a) neprekidna u tački $x=0$;

b) ima izvod u tački $x=0$;

c) ima neprekidan izvod u tački $x=0$?

Rješenje:

a)

$$0 \leq \left|x^{\alpha} \cdot \sin\frac{1}{x}\right| \leq |x^{\alpha}| \leq |x|^{\alpha}$$

Kako je $\lim_{x\to 0} |x|^{\alpha} = 0$, za $\alpha > 0$, to imamo na osnovu theoreme o uklještenju da je i:

$$\lim_{x\to 0} \left|x^{\alpha} \cdot \sin\frac{1}{x}\right| = 0, \quad \boxed{\alpha > 0}$$

$\Rightarrow$ $f(x)$ neprekidna u tački $x=0$ za $\boxed{\alpha > 0}$

b) $$f'(0) = \lim_{x\to 0} \frac{x^{\alpha} \cdot \sin\frac{1}{x}}{x} = \lim_{x\to 0}\left(x^{\alpha-1} \cdot \sin\frac{1}{x}\right) = 0, [\alpha$$

c) $$f'(x) = \alpha x^{\alpha-1} \cdot \sin\frac{1}{x} + x^{\alpha} \cdot \cos\frac{1}{x} \cdot \frac{-1}{x^2}, \quad x \neq 0$$

$$f'(x) = \alpha \cdot x^{\alpha-1} \cdot \sin\frac{1}{x} - x^{\alpha-2} \cdot \cos\frac{1}{x}$$

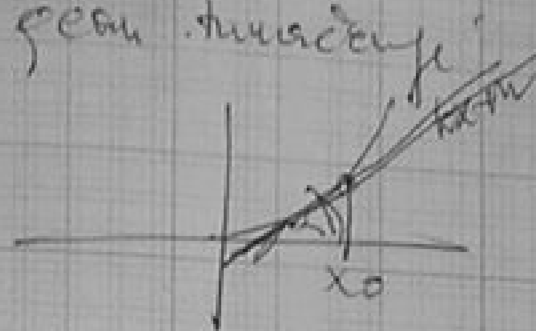
$$f'(x) = \begin{cases} \alpha \cdot x^{\alpha-1} \cdot \sin\frac{1}{x} - x^{\alpha-2} \cdot \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

* uslov neprekidnosti: $L_1 = L_2 = L = \lim_{x\to 0}\left(\alpha \cdot x^{\alpha-1} \sin\frac{1}{x} - x^{\alpha-2}\right.$

K: $\exists\, L = 0$ za $\cap \begin{cases} \alpha - 1 > 0 \\ \alpha - 2 > 0 \end{cases} \Rightarrow$ za $\boxed{\alpha > 2}$

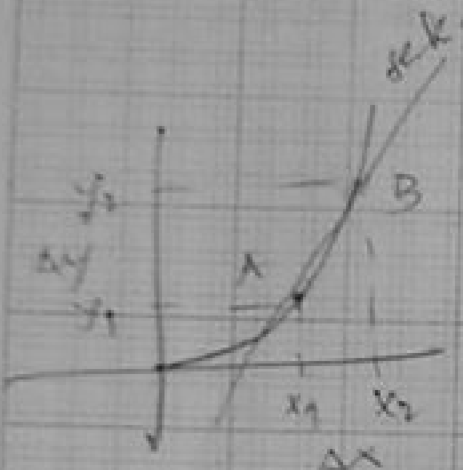
253.1. Polazeći od definicije izvoda funkcije, naći izvod funkcije $y = x^2$ (za bilo koji x).

Rješenje:



$y'(x_0) = \operatorname{tg}\alpha = k$

$$y \equiv f(x) = x^2$$



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2\cdot\Delta x\cdot x + (\Delta x)^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x)$$

$\Delta x \to 0$
$B \to A$
sek $\to$ tang.

$$\Rightarrow \boxed{f'(x) = 2x, \ \forall x}$$

254. Odrediti po definiciji izvoda, izvode funkcija:

a) $f(x) = \sqrt{x+a}$, u tački $x \ (\geq -a)$

b) $f(x) = (x-1)(x-2)^2(x-3)^3$ za one vrijednosti x za koje je $f(x) = 0$

Rješenje:

$$f(x) = 0 \Rightarrow x_1 = 1, \ x_2 = 2, \ x_3 = 3$$

$(1+\Delta x-$

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - \overset{0}{f(1)}}{\Delta x} = \lim_{\Delta x \to 0} \frac{(\cancel{1+\Delta x-1})(1+\Delta x-2)^2}{\cancel{\Delta x}}$$

$$f'(1) = \lim_{\Delta x \to 0}\left[(\Delta x - 1)^2 \cdot (\Delta x - 2)^3\right] = -1 \cdot 8 = -8$$

ii) $$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(1) = \lim_{x \to 1} \frac{\cancel{(x-1)}(x-2)^2(x-3)^3 - 0}{\cancel{x-1}} = (1-2)^2 \cdot (1-3)^3 = -8$$

c) $f(x) = x^2 \cdot \sin(x-2)$ u $x = 2$

$$f'(2) = \lim_{x \to 2} \frac{x^2 \cdot \sin(x-2) - 4 \cdot \cancel{\sin 0}^{\,0}}{x-2} = \lim_{x \to 2}\left[x^2 \cdot \frac{\sin(x-2)}{x-2}\right] =$$

ii)

$$f'(2) = \lim_{\Delta x \to 0} \frac{(2+\Delta x)^2 \sin(\cancel{2} + \Delta x - \cancel{2}) - \cancel{2^2 \cdot \sin(2-2)}^{\,0}}{\Delta x} =$$

$$= \lim_{\Delta x \to 0}\left[(\Delta x + 2)^2 \cdot \frac{\sin \Delta x}{\Delta x}\right] = 4$$

d) $f(x) = x + (x-1) \arcsin \sqrt{\frac{x}{x+1}}$, $x = 1$

$$f'(1) = \lim_{x \to 1} \frac{x + (x-1) \arcsin \sqrt{\frac{x}{x+1}} - 1}{x-1} =$$

$$= \lim_{x \to 1}\left[1 + \arcsin \sqrt{\frac{x}{x+1}}\right] = 1 + \arcsin \frac{1}{\sqrt{2}} =$$

$$= 1 + \arcsin \frac{\sqrt{2}}{2} = 1 + \frac{\pi}{4}$$

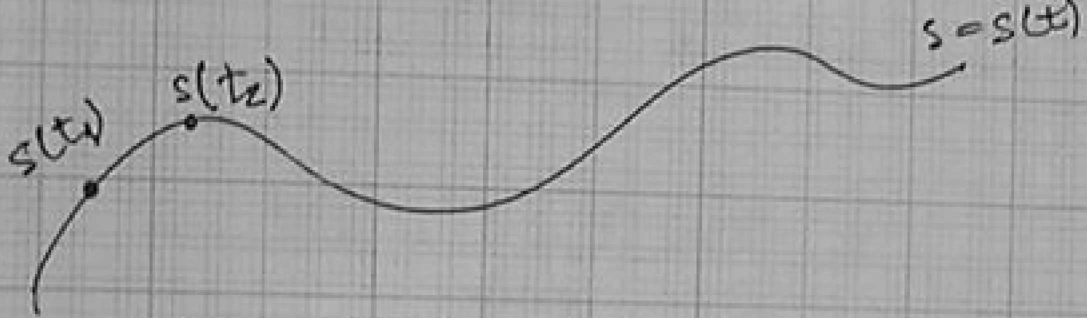
253. 3.

Neka se tačka kreće po nekoj putanji, sa zakonom puta $s = 3t^3 - 2t^2 + 1$, gdje je s pređeni put mjeren u cm u momentu t, a t je vrijeme mjereno u sekundama.

1° Naći srednju brzinu na razmaku $< t_1, t_2 >$ za $t_1 = 1$ i t_2

2° Naći brzinu promjene date funkcije za $t = 1$.

Rješenje



$$\Delta s = s(t_2) - s(t_1)$$

$$\boxed{v_{sr} = \frac{\Delta s}{\Delta t}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{3t_2^3 - 2t_2^2 + \cancel{1} - 3t_1^3 + 2t_1^2 - \cancel{1}}{t_2 - t_1}$$

1° $$v_{sr} = \frac{3(t_2^3 - t_1^3) - 2(t_2^2 - t_1^2)}{t_2 - t_1} \Rightarrow$$

$$v_{sr} = 3(t_2^2 + t_1 t_2 + t_1^2) - 2(t_2 + t_1) =$$

$$= 3(1{,}21 + 1{,}1 + 1) - 2(1{,}1 + 1) = 9{,}93 - 4{,}2$$

$$\Rightarrow \boxed{v_{sr} = 5{,}73}$$

* Brzina u bilo kojem trenutku se računa kao: $v(t) = \frac{d}{d}$

2° $v(t) = \frac{ds(t)}{dt} = 9t^2 - 4t$

$v(t_1) = 9 \cdot t_1^2 - 4 \cdot t_1 = 9 - 4 = 5$, $\boxed{v(t_1) = 5}$

255. b. Pokazati da funkcija

$$f(x) = \begin{cases} x^2, & x \text{ racionalno} \\ 0, & x \text{ iracionalno} \end{cases}$$

ima izvod samo u tački $x = 0$.

Rješenje

Dokažimo najprije da je funkcija $f(x)$ neprekidna samo u tački $x =$

* x racionalno: $\lim_{x \to a} f(x) = \lim_{x \to a} x^2 = a^2 = L_1$

* x iracionalno: $\lim_{x \to a} f(x) = \lim_{x \to a} 0 = 0 = L_2$

Pošto je $L_1 \neq L_2$ $\forall a (\neq 0) \Rightarrow$ funkcija $f(x)$ je prekidna u svakoj tački $x (\neq 0)$. $\Rightarrow$ Ima smisla tražiti izvod $f(x)$ samo u onim tačkama u kojima je ta fc. neprekidna u našem slučaju u tački $x = 0$:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - \overbrace{f(0)}^{0}}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x}, \quad 0 \leq \left| \frac{f(x)}{x} \right| \leq |x|$$

$$\Rightarrow f'(0) = \lim_{x \to 0} \frac{f(x)}{x} = 0.$$

258. Naći izvode složenih funkcija:

a) $y = \dfrac{x \cdot \sin x + \cos x}{x \cdot \cos x - \sin x}$

b) $y = \dfrac{2x}{1-x^2}$

c) $y = \dfrac{1 + x - x^2}{1 - x + x^2}$

d) $y = \log_a \dfrac{x}{1-x^2}$

Rešenje:

b) $\left(\dfrac{u}{v}\right)' = \dfrac{u' \cdot v - u \cdot v'}{v^2}, \; v \neq 0$

$y = \dfrac{2x}{1-x^2}, \; x \neq \pm 1$

$y' = \dfrac{2 \cdot (1-x^2) + 2x \cdot 2x}{(1-x^2)^2} = \cancel{\dfrac{2}{1-x^2}} \; \dfrac{2 - 2x^2 + 4x^2}{(1-x^2)^2}$

$= \dfrac{2\,(1+x^2)}{(1-x^2)^2}$

d) $y = \log_a \dfrac{x}{1-x^2}, \; \cap \begin{cases} \dfrac{x}{1-x^2} > 0 \\ 1 - x^2 \neq 0 \end{cases}$

$y' = \dfrac{\log_a e}{\dfrac{x}{\cancel{1-x^2}}} \cdot \dfrac{1 - x^2 - x \cdot (-2x)}{(1-x^2)^{\cancel{2}}} = \dfrac{1 - x^2 + 2x^2}{x\,(1-x^2)} \cdot \log$

$y' = \dfrac{1 + x^2}{x\,(1-x^2)} \cdot \log_a e$

252. Naći izvode složenih funkcija:

a) $y = \sin(\cos^2 x) \cdot \cos(\sin^2 x)$

b) $y = x^{a^a} + a^{x^a} + a^{a^x}$; $(a>0 ; x>0)$

c) $y = \ln[\ln^2(\ln^3 x)]$

d) $y = \arccos(\cos(\arccos x))$

Rješenje:

a) $y = f_1(x) \cdot f_2(x)$, $f_1 = \sin(\cos^2 x)$, $f_2 = \cos(\sin^2 x)$

$y' = f_1'(x) \cdot f_2(x) + f_1(x) \cdot f_2'(x)$

$f_1'(x) = -\cos(\cos^2 x) \cdot \underbrace{2\cos x \cdot \sin x}_{\sin 2x} = -\sin 2x \cdot \cos(co$

$\cancel{f_2'(x) = \cos(\sin^2 x)}$

$f_2'(x) = -\sin(\sin^2 x) \cdot \underbrace{2\sin x \cdot \cos x}_{\sin 2x} = -\sin 2x \cdot \sin(s$

$y' = -\sin 2x \cdot \cos(\cos^2 x) \cdot \cos(\sin^2 x) - \sin(\cos^2 x) \cdot \sin 2x \cdot s$

$y' = -\sin 2x \left(\cos(\cos^2 x)\cos(\sin^2 x) + \sin(\cos^2 x)\sin(s \right.$

$= -\sin 2x \cdot \cos(\cos^2 x - \sin 2x)$

b) $(a^x)' = a^x \cdot \ln a$

$y = x^{a^a} + a^{x^a} + a^{a^x}$

$y' = a^a \cdot x^{a^a - 1} + \cancel{a}\, a^{x^a} \cdot \ln a \cdot a \cdot x^{a-1} +$

$+ a^{a^x} \cdot \cancel{a^x \cdot \ln a}\ \ln a \cdot a^x \cdot \ln a$