→ TUTORIJAL 11 ←

Neodređeni integrali

pravilo: $\boxed{\int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + C \quad , u(x) \neq 0}$

Fatkić M.

433) a) $\int \frac{3dx}{4+2x}$

$u(x) = 4 + 2x = 2(2+x) \neq 0$

$du = 2\,dx$

$$\int \frac{3dx}{2(2+x)} = \frac{3}{2}\int \frac{dx}{2+x} = \frac{3}{2}\int \frac{d(2+x)}{(2+x)} = \frac{3}{2}\ln|2+x| + C$$

b) $\int \frac{x\,dx}{x^2+1} = \left| \begin{array}{l} u(x) = x^2+1 \neq 0 \\ du = 2x\,dx \\ \Rightarrow x\,dx = \frac{du}{2} \end{array} \right| = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} = \frac{1}{2}\ln(x^2+1) + C$

integracija metodom zamjene

434.) a) $\int \frac{dx}{\sqrt{k^2-x^2}} \overset{k>0}{=} \left| \begin{array}{l} \text{želimo svesti na oblik} \\ \int \frac{dx}{\sqrt{1-x^2}} \Rightarrow \begin{array}{l} x = k\cdot t \\ dx = k\,dt \\ \Rightarrow t = \frac{x}{k} \end{array} \end{array} \right| =$

$$= \int \frac{k\,dt}{\sqrt{k^2-k^2t^2}} = \int \frac{\cancel{k}\,dt}{\cancel{k}\sqrt{1-t^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C =$$

$$= \boxed{\arcsin \frac{x}{k} + C}$$

b) $\int \frac{dx}{\sqrt{5+4x-x^2}} = \left| \begin{array}{l} 5+4x-x^2 = \\ = 3^2 - (x-2)^2 \\ x-2 = t \\ dx = dt \end{array} \right| = \int \frac{dt}{\sqrt{3^2-t^2}} =$

$$= \arcsin \frac{t}{3} + C = \boxed{\arcsin \frac{x-2}{3} + C}$$

+34.10* $\int \frac{dx}{\sin x} = \left| \begin{array}{l} \mathrm{tg}\frac{x}{2} = t \\ dt = \frac{1}{2}\frac{dx}{\cos^2\frac{x}{2}} \end{array} \right| =$

$$= \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\mathrm{tg}\frac{x}{2}\cdot\cos^2\frac{x}{2}} =$$

$$= \frac{1}{\cancel{2}}\int \frac{\cancel{2}dt}{t} = \int \frac{dt}{t} = \ln|t| + C = \boxed{\ln\left|\mathrm{tg}\frac{x}{2}\right| + C}$$

$$\int \frac{dx}{\cos x} = \int \frac{d(x+\frac{\pi}{2})}{\cos(x+\frac{\pi}{2})} = \int \frac{dy}{\sin y} =$$

$$= \ln\left|\mathrm{tg}\frac{y}{2}\right| + C = \boxed{\ln\left|\mathrm{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C}$$

34.11* $\int \frac{dx}{\mathrm{sh}x}$

$$\mathrm{sh}x = \frac{e^x - e^{-x}}{2} \qquad \mathrm{ch}x = \frac{e^x + e^{-x}}{2}$$

$$\int \frac{dx}{\mathrm{sh}x} = 2\int \frac{dx}{e^x - e^{-x}} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| =$$

$$dx = \frac{dt}{e^x} = \frac{dt}{t}$$

$$= 2\int \frac{dt/t}{t - \frac{1}{t}} =$$

$$= 2\int \frac{dt}{t(t-\frac{1}{t})} = 2\int \frac{dt}{t^2-1} = \cancel{2}\cdot\frac{1}{\cancel{2}}\ln\left|\frac{t-1}{t+1}\right| + C =$$

$$= \ln\left|\frac{e^x-1}{e^x+1}\right| + C = \ln\left|\frac{\cancel{e^{\frac{x}{2}}}(e^{\frac{x}{2}} - e^{-\frac{x}{2}})}{\cancel{e^{\frac{x}{2}}}(e^{\frac{x}{2}} + e^{-\frac{x}{2}})}\right| + C =$$

$$= \ln\left|\frac{\frac{e^{\frac{x}{2}} - e^{-\frac{x}{2}}}{2}\cdot\cancel{2}}{\frac{e^{\frac{x}{2}} + e^{-\frac{x}{2}}}{2}\cdot\cancel{2}}\right| + C = \ln\left|\frac{\mathrm{sh}\frac{x}{2}}{\mathrm{ch}\frac{x}{2}}\right| + C =$$

$$= \boxed{\ln\left|\mathrm{th}\frac{x}{2}\right| + C}$$

435*
$$\int \frac{x^9 dx}{(1+x^5)^3} = \left| \begin{array}{l} 1+x^5 = t \Rightarrow x^5 = t-1 \\ 5x^4 dx = dt \\ \Rightarrow x^9 dx = x^5 \, x^4 dx = \\ = (t-1) \cdot \frac{dt}{5} \end{array} \right| =$$

$$= \int \frac{(t-1)\, dt/5}{t^3} = \frac{1}{5} \int \frac{(t-1)\,dt}{t^3} =$$

$$= \frac{1}{5}\left[\int \frac{t\,dt}{t^3} - \int \frac{dt}{t^3} \right] = \frac{1}{5}\left[\int \frac{dt}{t^2} - \int \frac{dt}{t^3} \right] =$$

$$= \frac{1}{5}\left[\frac{t^{-2+1}}{-2+1} - \frac{t^{-3+1}}{-3+1} \right] + C = \frac{1}{5}\left[\frac{t^{-1}}{-1} + \frac{t^{-2}}{2} \right] + C =$$

$$= \frac{1}{5}\left[\frac{1}{2t^2} - \frac{1}{t} \right] + C = \frac{1-2t}{10t^2} + C =$$

$$= \boxed{\frac{1-2(1+x^5)}{10(1+x^5)^2} + C}$$

parcijalna integracija: $\boxed{\int u \cdot dv = u \cdot v - \int v \cdot du}$

38.*
a) $$\underbrace{\int e^{ax} \cos bx \, dx}_{I} = \left| \begin{array}{ll} u = \cos bx & dv = e^{ax} dx \\ du = -b \sin bx \, dx & v = \int dv = \frac{1}{a} e^{ax} \end{array} \right| =$$

$$= \cos bx \cdot \frac{1}{a} e^{ax} + \int \frac{1}{a} e^{ax} \, b \sin bx \, dx =$$

$$= \frac{1}{a} \cos bx \, e^{ax} + \frac{b}{a} \underbrace{\int e^{ax} \sin bx \, dx}$$

$$\begin{array}{ll} u = \sin bx & dv = e^{ax} dx \\ du = b \cos bx \, dx & v = \int dv = \frac{1}{a} e^{ax} \end{array}$$

$$= \frac{1}{a} \cos bx \, e^{ax} + \frac{b}{a}\left[\sin bx \cdot \frac{1}{a} e^{ax} - \frac{b}{a} \underbrace{\int e^{ax} \cos bx \, dx}_{I} \right]$$

$$I = \frac{1}{a} \cos bx \, e^{ax} + \frac{b}{a^2} e^{ax} \sin bx - \frac{b^2}{a^2} I$$

$$I \left(1 + \frac{b^2}{a^2}\right) = \frac{1}{a} e^{ax} \cos bx + \frac{b}{a^2} e^{ax} \sin bx$$

$$I \cdot \frac{a^2+b^2}{a^2} = \frac{1}{a} e^{ax}\cos bx + \frac{b}{a^2} e^{ax}\sin bx$$

$$I = \frac{\cancel{a^2}}{a^2+b^2} e^{ax} \cdot \frac{1}{\cancel{a}} \cos bx + \frac{\cancel{a^2}}{a^2+b^2} \cdot \frac{b}{\cancel{a^2}} e^{ax} \sin bx$$

$$\boxed{I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)}$$

integracija racionalnih fcija

443.* $\int \frac{x^5+x^4-8}{x^3-4x}dx = I$

$$(x^5+x^4-8):(x^3-4x) = x^2+x+4$$

$$\underline{-x^5 \underset{+}{-} 4x^3}$$

$$= \quad x^4+4x^3-8$$

$$\underline{-x^4 \underset{+}{-} 4x^2}$$

$$= \quad 4x^3+4x^2-8$$

$$\underline{-4x^3 \underset{+}{-} 16x}$$

$$= \quad 4x^2+16x-8$$

$$\Rightarrow I = \int\left[x^2+x+4+\frac{4x^2+16x-8}{x^3-4x}\right]dx =$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+4x+\int\frac{4x^2+16x-8}{x^3-4x}dx =$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+4x+4\underbrace{\int\frac{x^2+4x-2}{x^3-4x}dx}_{I_1} + C_2$$

$$\frac{x^2+4x-2}{x(x-2)(x+2)} = \frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+2}$$

$$x^2+4x-2 \equiv A(x^2-4)+B(x^2+2x)+C(x^2-2x)$$

$$x^2+4x-2 \equiv Ax^2-4A+Bx^2+2Bx+Cx^2-2Cx$$

$$\equiv \underbrace{(A+B+C)}_{=1}x^2+\underbrace{(2B-2C)}_{=4}x+\underbrace{(-4A)}_{=-2}$$

$A = 2 \Rightarrow A = +\frac{1}{2}$

$A + B + C = 1 \qquad B + C = 1 - A = 1 - \frac{1}{2} = \frac{1}{2}$

$2(B - C) = 4 \Rightarrow B - C = 2$

$2B = 2 + \frac{1}{2} = \frac{4+1}{2} = \frac{5}{2}$

$B = \frac{5}{4}$

$B - C = 2 \Rightarrow C = B - 2 = \frac{5}{4} - 2 = \frac{5-4}{4}$

$C = \frac{1}{4}$

$$I_1 = \int \frac{x^2+4x-2}{x^3-4x}dx = \int \left(+\frac{1}{2}\right)\frac{dx}{x} + \frac{5}{4}\int\frac{dx}{x-2} + \frac{1}{4}\int\frac{dx}{x+2} =$$

$$= +\frac{1}{2}\ln|x| + \frac{5}{4}\ln|x-2| + \frac{1}{4}\ln|x+2| + C_1$$

$$\boxed{I = \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2\ln|x| + 5\ln|x-2| + \ln|x+2| + C}$$

$C = C_1 + C_2$

448.* $\displaystyle\int \frac{dx}{x^4+2x^3+2x^2+2x+1} = I$

$x^4 + 2x^3 + 2x^2 + 2x + 1 = (x^2+1)^2 + 2x(x^2+1) =$

$= (x^2+1)(x^2+1+2x) = (x^2+1)(x+1)^2$

$$\frac{1}{(x^2+1)(x+1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \Big/ (x^2+1)(x+1)^2$$

$$1 \equiv (Ax+B)(x^2+2x+1) + \overbrace{C(x^2+1)}^{Cx^2+C}(x+1) + D(x^2+1)$$

$\equiv Ax^3 + Bx^2 + 2Ax^2 + 2Bx + Ax + B + Cx^3 + Cx + Cx^2 + C + Dx^2 + D$

$\equiv \underbrace{(A+C)}_{=0}x^3 + \underbrace{(B+2A+C+D)}_{=0}x^2 + \underbrace{(2B+A+C)}_{=0}x + \underbrace{B+C+D}_{=1}$

$A+C=0 \Rightarrow A=-C$

$2A+B+C+D=0$

$A+2B+C=0$

$B+C+D=1$

$-2C+B+C+D=0$

$-\cancel{C}+2B+\cancel{C}=0 \Rightarrow B=0$

$0+C+D=1 \Rightarrow C=1-D$

$-2C+0+C+D=0$

$D-C=0$

$D-1+D=0$

$2D=1$

$D=\frac{1}{2} \Rightarrow C=\frac{1}{2}$

$A=-\frac{1}{2}$

$$I=\int\frac{-\frac{1}{2}x\,dx}{x^2+1}+\int\frac{\frac{1}{2}dx}{x+1}+\int\frac{\frac{1}{2}dx}{(x+1)^2}=$$

$$=-\frac{1}{2}\underbrace{\int\frac{x\,dx}{x^2+1}}+\frac{1}{2}\underbrace{\int\frac{dx}{x+1}}+\frac{1}{2}\int\frac{dx}{(x+1)^2}$$

$x^2+1=t$, $2x\,dx=dt$, $x\,dx=\frac{dt}{2}$

$x+1=z$, $dx=dz$

$x+1=y$, $dx=dy$

$$=-\frac{1}{2}\frac{1}{2}\int\frac{dt}{t}+\frac{1}{2}\int\frac{dz}{z}+\frac{1}{2}\int\frac{dy}{y^2}=$$

$$=-\frac{1}{4}\ln|x^2+1|+\frac{1}{2}\ln|x+1|+\frac{1}{2}\cdot\frac{y^{-2+1}}{-2+1}+C=$$

$$=-\frac{1}{4}\ln(x^2+1)+\frac{1}{4}\,2\ln|x+1|-\frac{1}{2}\frac{1}{x+1}+C=$$

$$\boxed{\frac{1}{4}\ln\frac{(x+1)^2}{x^2+1}-\frac{1}{2}\frac{1}{x+1}+C}$$

Integracija iracionalnih fcija

1° $R(x, \sqrt[n]{ax+b}) \Rightarrow$ smjena $t^n = ax+b$

2° $R(x, \sqrt[n]{\frac{ax+b}{cx+d}}) \Rightarrow$ smjena $t^n = \frac{ax+b}{cx+d}$

$\uparrow$ racionalna fcija

3° $R(x, \sqrt{ax^2+bx+c}) \Rightarrow$ $t + x\sqrt{a}$, $a>0$

$tx + \sqrt{c}$, $c>0$

$t(x-x_0)$, x_0 - jednostruki korijen trinoma ax^2+bx+c

450.

d) $\int \frac{\sqrt[4]{x}}{\sqrt[3]{x}+\sqrt{x}}\,dx = \left| \begin{array}{l} x = t^{12} \\ dx = 12t^{11}dt \end{array} \right| =$

$= \int \frac{t^3 \cdot 12t^{11}dt}{t^4+t^6} = 12\int \frac{t^{14}dt}{t^4+t^6} = 12\int \frac{t^{14}dt}{t^4(1+t^2)} = 12\int \frac{t^{10}dt}{1+t^2}$

$t^{10} : (t^2+1) = t^8 - t^6 + t^4 - t^2 + 1$

$\underline{-t^{10} \pm t^8}$

$= \quad -t^8$

$\underline{\mp t^8 \mp t^6}$

t^6

$\underline{-t^6 - t^4}$

$-t^4$

$\underline{\mp t^4 \mp t^2}$

t^2

$\underline{-t^2 + 1}$

-1

$I = 12\int \left[t^8 - t^6 + t^4 - t^2 + 1 - \frac{1}{1+t^2}\right]dt =$

$= 12\left\{\frac{t^9}{9} - \frac{t^7}{7} + \frac{t^5}{5} - \frac{t^3}{3} + t - \operatorname{arctg} t\right\} + C =$

$= \boxed{\frac{4}{3}\sqrt[4]{x^3} - \frac{12}{7}\sqrt[12]{x^7} + \frac{12}{5}\sqrt[12]{x^5} - 4\sqrt[4]{x} + 12\sqrt[12]{x} - \operatorname{arctg}\sqrt[12]{x} + C}$

Binomni integral $\int x^p (ax^q+b)^r dx$

uslov:

1° $\frac{p+1}{q}$ cio broj $\Rightarrow$ smjena $ax^q+b=t^s$

↳ da ↑

↓ ne

2° $\frac{p+1}{q}+r$ cio broj $\Rightarrow$ smjena $ax^q+b=t^s\cdot x^q$

+54.b.* $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = \int x^{-1/2}(1+x^{1/4})^{1/3}dx$

$p=-\frac{1}{2},\ a=1,\ b=1,\ q=\frac{1}{4},\ r=\frac{1}{3}$

$\frac{p+1}{q}=\frac{-\frac{1}{2}+1}{\frac{1}{4}}=\frac{\frac{1}{2}}{\frac{1}{4}}=2$ ← cio broj $\Rightarrow$ smjena:

$1+x^{1/4}=t^3$

$x^{1/4}=t^3-1$

$x=(t^3-1)^4$

$dx=4(t^3-1)^3\cdot 3t^2\,dt=$

$=12t^2(t^3-1)^3dt$

$I=\int\left[(t^3-1)^4\right]^{-1/2}\cdot t\cdot 12t^2(t^3-1)^3dt=$ $\quad(-2)$

$=12\int t^3(t^3-1)dt=12\int(t^6-t^3)dt=$

$=12\left[\frac{t^7}{7}-\frac{t^4}{4}\right]+C=$

$=\boxed{\frac{12}{7}\sqrt[3]{(1+\sqrt[4]{x})^7}-3\cdot\sqrt[3]{(1+\sqrt[4]{x})^4}+C}$

Integracija trigonometrijskih fcija

1° $R(\sin x, \cos x) \Rightarrow t = \operatorname{tg}\frac{x}{2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$

$$dx = \frac{2dt}{1+t^2}$$

2° ako je $R(-\sin x, \cos x) = -R(\sin x, \cos x) \Rightarrow \cos x = t$

3° ako je $R(\sin x, -\cos x) = -R(\sin x, \cos x) \Rightarrow \sin x = t$

4° ako je $R(-\sin x, -\cos x) = +R(\sin x, \cos x) \Rightarrow \operatorname{tg} x = t$

461* $\int \frac{3\sin x + 2\cos x}{2\sin x + 3\cos x}\,dx$

$R(-\sin x, -\cos x) = R(\sin x, \cos x) \Rightarrow$ smjena $\operatorname{tg} x = t$

$\frac{dx}{\cos^2 x} = dt$

$\sin^2 x + \cos^2 x = 1 \;/ : \cos^2 x$

$\operatorname{tg}^2 x + 1 = \frac{1}{\cos^2 x}$

$\cos^2 x = \frac{1}{\operatorname{tg}^2 x + 1} \Rightarrow dx(\operatorname{tg}^2 x + 1) = dt$

$dx(t^2+1) = dt$

$dx = \frac{dt}{t^2+1}$

$$I = \int \frac{3\operatorname{tg} x + 2}{2\operatorname{tg} x + 3}\,dx = \int \frac{3t+2}{2t+3}\,\frac{dt}{t^2+1}$$

$$\frac{3t+2}{(2t+3)(t^2+1)} = \frac{A}{2t+3} + \frac{Bt+C}{t^2+1} \;\Big/ \cdot (2t+3)(t^2+1)$$

$$3t+2 = A(t^2+1) + (Bt+C)(2t+3)$$

$$= At^2 + A + 2Bt^2 + 2Ct + 3Bt + 3C$$

$$= \underbrace{(A+2B)}_{=0}t^2 + \underbrace{(2C+3B)}_{=3}t + \underbrace{A+3C}_{=2}$$

$$A+2B=0 \Rightarrow A=-2B$$
$$2C+3B=3$$
$$A+3C=2$$

$$2C+3B=3$$
$$-2B+3C=2$$

$$3C=2+2B$$
$$C=\frac{2+2B}{3} \Rightarrow 2\,\frac{2+2B}{3}+3B=3 \,/\cdot 3$$

$$4+4B+9B=9$$
$$13B=5$$
$$\boxed{B=\frac{5}{13}}$$

$$\boxed{C}=\frac{2+2B}{3}=\frac{2+\frac{10}{13}}{3}=\frac{36}{39}\begin{matrix}:3\\:3\end{matrix}=\boxed{\frac{12}{13}}$$

$$\boxed{A}=-2B=\boxed{-\frac{10}{13}}$$

$$t^2+1=y$$
$$2t\,dt=dy \Rightarrow t\,dt=\frac{dy}{2} \Rightarrow \frac{1}{2}\int\frac{dy}{y}$$

$$I=\int\left(-\frac{10}{13}\right)\frac{dt}{2t+3}+\frac{5}{13}\int\frac{t\,dt}{t^2+1}+\frac{12}{13}\int\frac{dt}{t^2+1}=$$

$$=-\frac{5}{13}\cdot\ln|2t+3|+\frac{5}{13}\cdot\frac{1}{2}\ln|t^2+1|+\frac{12}{13}\operatorname{arctg}t+C=$$

$$=-\frac{5}{13}\ln|2t+3|+\frac{5}{13}\cdot\ln\sqrt{t^2+1}+\frac{12}{13}\operatorname{arctg}t+C=$$

$$=-\frac{5}{13}\ln|2\operatorname{tg}x+3|+\frac{5}{13}\ln\sqrt{\frac{1}{\cos^2x}}+\frac{12}{13}\operatorname{arctg}(\operatorname{tg}x)+C=$$

$$=-\frac{5}{13}\ln|2\operatorname{tg}x+3|+\frac{5}{13}\ln\frac{1}{|\cos x|}+\frac{12}{13}x+C=$$

$$=-\frac{5}{13}\ln\left|\frac{2\sin x+3\cos x}{\cos x}\right|+\frac{5}{13}\ln\frac{1}{|\cos x|}+\frac{12}{13}x+C=$$

$$=\boxed{-\frac{5}{13}\ln|2\sin x+3\cos x|+\frac{12}{13}x+C}$$