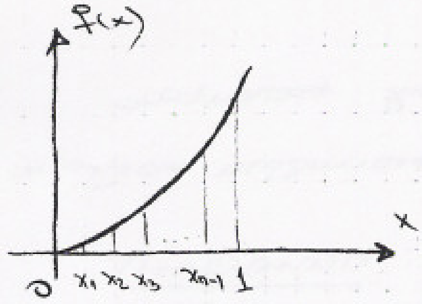
→ TUTORIJAL 12 ←

Određeni i višestruki integrali

Određeni integral

1.2. Izračunati površinu krivolinijskog trokuta ograničenog grafikom fcje $f(x)=x^2$, segmentom $[0,1]$ na x-osi, te ordinatama bačaka grafika čije su apscise $x=0$ i $x=1$.



Podijelimo segment $[0,1]$ na n jednakih podsegmenata $\tau_i = \left[\frac{i-1}{n}, \frac{i}{n}\right]$, $(i=\overline{1,n})$. Fcja $f(x)=x^2$ je M↑ na $[0,1]$. Pošto je $f(x)$ nep. fcja, to se donje(gornje) međe fcje podudaraju se najmanjim(najvećim) vrijednostima fcje na odgovarajućim podsegmentima τ_i, tj.:

$$m_1 = f(0) = 0^2 = 0 \qquad M_1 = f\left(\frac{1}{n}\right) = \frac{1}{n^2}$$

$$m_2 = f\left(\frac{1}{n}\right) = \frac{1}{n^2} \qquad M_2 = f\left(\frac{2}{n}\right) = \frac{4}{n^2}$$

$$\vdots \qquad\qquad \vdots$$

$$m_i = f\left(\frac{i-1}{n}\right) = \frac{(i-1)^2}{n^2} \qquad M_i = f\left(\frac{i}{n}\right) = \frac{i^2}{n^2}$$

$$\vdots \qquad\qquad \vdots$$

$$m_n = f\left(\frac{n-1}{n}\right)^2 = \frac{(n-1)^2}{n^2} \qquad M_n = f\left(\frac{n}{n}\right) = 1$$

Pošto smo segment $[0,1]$ podijelili na n jednakih dijelova, a je dužina svake ćelije τ_i jednaka

$$\sigma_i = \frac{i}{n} - \frac{i-1}{n} = \frac{i-i+1}{n} = \frac{1}{n}$$

Prema tome, površina stepenastog poligona upisanog u krivolinijski trokut odozgo omeđenog grafikom fcje $f(x)$, sa strane pravcima $x=0$ i $x=1$, a odozdo segmentom $[0,1]$ iznosi:

$$\sum_{i=1}^{n} m_i \sigma_i = \sum_{i=1}^{n} \frac{(i-1)^2}{n^2} \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 =$$

$$= \frac{1}{n^3}\left(1+2^2+3^2+\dots+(n-1)^2\right)$$

Iz $\quad 1+2^2+3^2+\dots+n^2=\frac{n(n+1)(2n+1)}{6}$ $\Leftarrow$ može se dokazati matematičkom indukcijom

$$\Rightarrow \quad 1+2^2+3^2+\dots+(n-1)^2=\frac{(n-1)\cdot n(2(n-1)+1)}{6}=$$
$$=\frac{n(n-1)(2n-1)}{6}$$

Dakle, $\sum_{i=1}^{n} m_i \delta_i = \frac{n(n-1)(2n-1)}{6n^3}$

Površina oko navedenog krivolinijskog trokuta, opisanog stepenastog poligona za proizvoljnu podjelu τ iznosi:

$$\sum_{i=1}^{n} M_i\delta_i = \sum_{i=1}^{n}\frac{i^2}{n^2}\cdot\frac{1}{n}=\frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

Prelaskom na graničnu vrijednost dobijamo

$$\lim_{n\to+\infty}\sum_{i=1}^{n} m_i\delta_i = \lim_{n\to+\infty}\frac{n(n-1)(2n-1):n^3}{6n^3\ :n^3}=$$
$$=\lim_{n\to+\infty}\frac{1\left(1-\overset{\to 0}{\frac{1}{n}}\right)\left(2-\overset{\to 0}{\frac{1}{n}}\right)}{6}=\frac{2}{6}=\frac{1}{3}$$

$$\lim_{n\to+\infty}\sum_{i=1}^{n} M_i\delta_i = \lim_{n\to\infty}\frac{n(n+1)(2n+1):n^3}{6n^3\ :n^3}=$$
$$=\lim_{n\to+\infty}\frac{1\left(1+\overset{\to 0}{\frac{1}{n}}\right)\left(2+\overset{\to 0}{\frac{1}{n}}\right)}{6}=\frac{2}{6}=\frac{1}{3}$$

Pošto je $\lim_{n\to\infty}\sum_1^n m_i\delta_i=\lim_{n\to\infty}\sum_1^n M_i\delta_i = P \Rightarrow$ površina krivolinijskog trokuta iznosi $\boxed{P=\frac{1}{3}}$.

~~određeni integral~~

~~$\int_a^b x^2\,dx$, $0 \le a < b$, Rezult.~~

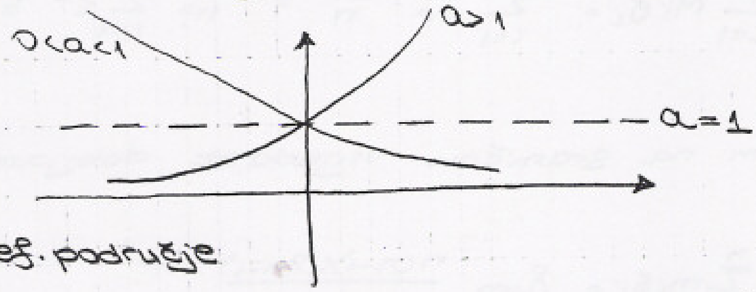
3: Funkcije koje su na segmentu $[a,b]$

1. neprekidne
2. monotone
3. ograničene (sa konačno mnogo tačaka prekida)

su integrabilne na tom segmentu.

4.1. Ispitati integrabilnost fcje: $f(x)=a^x$.

$f(x)=a^x$



$a>0,\ a\neq 1 \leftarrow$ def. područje

Ova fcja je neprekidna na proizvoljnom segmentu $[b,c] \subset (-\infty,+\infty)$, pa je i integrabilna na tom segmentu.

5.3. Procijeniti integrale

$$\boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)}$$

1° $\int_0^{\pi} \frac{dx}{3+\sin^2 x}$

$$0 \le \sin^2 x \le 1$$

$$3 \le 3+\sin^2 x \le 1+3$$

$$\underbrace{\frac{1}{4}}_{m} \le \frac{1}{3+\sin^2 x} \le \underbrace{\frac{1}{3}}_{M}$$

$$\Rightarrow \frac{\pi}{4} \le \int_0^{\pi} \frac{1}{3+\sin^2 x}\,dx \le \frac{\pi}{3}$$

Iz $1+2^2+3^2+..+n^2=\frac{n(n+1)(2n+1)}{6}$ ⟸ ⟹ može se dokazati matematičkom indukcijom

$$\Rightarrow 1+2^2+3^2+..+(n-1)^2=\frac{(n-1)\cdot n(2(n-1)+1)}{6}=$$

$$=\frac{n(n-1)(2n-1)}{6}$$

Dakle, $\sum_{i=1}^{n} m_i \delta_i = \frac{n(n-1)(2n-1)}{6n^3}$

Površina oko navedenog krivolinijskog trokuta opisanog stepenastog poligona za proizvoljnu podjelu τ iznosi:

$$\sum_{i=1}^{n} M_i \delta_i = \sum_{i=1}^{n} \frac{i^2}{n^2}\cdot\frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6n^3}.$$

Prelaskom na graničnu vrijedost dobijamo

$$\lim_{n\to+\infty}\sum_{i=1}^{n} m_i\delta_i = \lim_{n\to+\infty}\frac{n(n-1)(2n-1):n^3}{6n^3:n^3}=$$

$$=\lim_{n\to+\infty}\frac{1\,(1-\overset{\to 0}{\frac{1}{n}})(2-\overset{\to 0}{\frac{1}{n}})}{6}=\frac{2}{6}=\frac{1}{3}$$

$$\lim_{n\to+\infty}\sum_{i=1}^{n} M_i\delta_i = \lim_{n\to+\infty}\frac{n(n+1)(2n+1):n^3}{6n^3:n^3}=$$

$$=\lim_{n\to+\infty}\frac{1\,(1+\overset{\to 0}{\frac{1}{n}})(2+\overset{\to 0}{\frac{1}{n}})}{6}=\frac{2}{6}=\frac{1}{3}$$

Pošto je $\lim_{n\to+\infty}\sum_{1}^{n} m_i\delta_i = \lim_{n\to+\infty}\sum_{1}^{n} M_i\delta_i = P \Rightarrow$ površina krivolinijskog trokuta iznosi $\boxed{P=\frac{1}{3}}$.

$2^\circ \quad \int_0^{3\pi/2} \frac{dx}{3+\sin x}$

$-1 \leq \sin x \leq 1$ na $[0, 3\pi/2]$

$-1+3 \leq 3+\sin x \leq 1+3$

$$\frac{1}{4} \leq \frac{1}{3+\sin x} \leq \frac{1}{2}$$

$$\Rightarrow \frac{\frac{3\pi}{2}}{4} \leq \int_0^{3\pi/2} \frac{dx}{3+\sin x} \leq \frac{3\pi/2}{2}$$

$$\frac{3\pi}{8} \leq \int_0^{3\pi/2} \frac{dx}{3+\sin x} \leq \frac{3\pi}{4}$$

__Th__ (__o srednjoj vrijednosti__)

Neka su $f(x)$ i $g(x)$ integrabilne f-c.je na segmentu $[a,b]$, $m \leq f(x) \leq M$ na $[a,b]$, a $g(x)$ ne mijenja predznak u intervalu (a,b), tj. $g(x) \geq 0$ ili $g(x) \leq 0$. Tada je:

$$\int_a^b f(x)g(x)dx = \mu \int_a^b g(x)dx,$$

gdje je $m \leq \mu \leq M$ (m je donja, a M gornja međa funkcije f na $[a,b]$).

Neka je $g(x) = 1$. Ako je $f(x)$ nep. na $[a,b]$, tada je $f(c) = \mu$ za neki $c \in [a,b]$.

Za navedene uslove imamo

$$\boxed{\int_a^b f(x)dx = \underbrace{f(c)}_{\text{srednja vrijednost f-c.je } f(x) \text{ na segmentu } [a,b]}(b-a)}$$

6.3.)* Za koje b je srednja vrijednost f-je $f(x)=\ln x$ na segmentu $[1,b]$ jednaka srednjoj brzini mijenjanja f-je na tom segmentu? 7

Srednja vrijednost f-je:

$$\int_a^b f(x)dx = f(c)(b-a)$$

$$\Rightarrow f(c) = \frac{\int_a^b f(x)dx}{b-a}$$

$$\mu = f(c) = \frac{1}{b-1}\int_1^b \ln x dx = \frac{1}{b-1}\left[x\ln x - x\right]_1^b =$$

$$= \frac{1}{b-1}\left[b\ln b - b - 1\cancel{\ln 1}^{0} + 1\right] =$$

$$= \frac{1}{b-1}\left[b\ln b - b + 1\right].$$

Srednja brzina mijenjanja f-je:

$$\frac{\Delta f(x)}{\Delta x} = \frac{\ln b - \ln 1}{b-1} = \frac{\ln b}{b-1}$$

Uslov zadatka:

$$\mu = \frac{\Delta f}{\Delta x} \Longleftrightarrow \frac{b\ln b - b + 1}{b-1} = \frac{\ln b}{b-1},$$

$b \neq 1$ 8

$$b\ln b - b + 1 = \ln b$$

$$b\ln b - \ln b = b - 1$$

$$(b-1)\ln b = b-1$$

$$\ln b = 1$$

$$\boxed{b = e}$$

7.3.b. Naći graničnu vrijednost

$$\lim_{x\to+\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$$

$$\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

$$\lim_{x\to+\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}=\lim_{x\to+\infty}\frac{\frac{d}{dx}\left[\left(\int_0^x e^{t^2}dt\right)^2\right]}{\frac{d}{dx}\left(\int_0^x e^{2t^2}dt\right)}=$$

$$=\lim_{x\to+\infty}\frac{2\left[\int_0^x e^{t^2}dt\right]\cdot \cancel{e^{x^2}}}{e^{2x^2}}=2\lim_{x\to+\infty}\frac{\int_0^x e^{t^2}dt}{e^{x^2}}=$$

$$=2\lim_{x\to+\infty}\frac{\frac{d}{dx}\int_0^x e^{t^2}dt}{\frac{d}{dx}e^{x^2}}=\cancel{2}\lim_{x\to+\infty}\frac{\cancel{e^{x^2}}}{\cancel{e^{x^2}}\cdot\cancel{2}x}=$$

$$=\lim_{x\to+\infty}\frac{1}{x}=\boxed{0}$$

Newton-Leibnizova formula

$$\boxed{\int_a^b f(x)\,dx=F(x)\Big|_a^b=F(b)-F(a)}$$

8.1.i) $$\int_0^1\frac{x^2dx}{\sqrt{x^6+4}}=\left|\begin{array}{l}x^3=t\\3x^2dx=dt\\x^2dx=dt/3\\x=1\Rightarrow t=1\\x=0\Rightarrow t=0\end{array}\right|=\int_0^1\frac{dt/3}{\sqrt{t^2+2^2}}=$$

$$=\frac{1}{3}\int_0^1\frac{dt}{\sqrt{t^2+a^2}}=\frac{1}{3}\ln\left|t+\sqrt{t^2+2^2}\right|\Big|_0^1=\frac{1}{3}\ln\left|1+\sqrt{1+4}\right|-$$

$$-\frac{1}{3}\ln\left|0+\sqrt{4}\right|=\frac{1}{3}\left[\ln(1+\sqrt{5})-\ln 2\right]=\boxed{\frac{1}{3}\ln\frac{1+\sqrt{5}}{2}}$$

Parcijalna integracija

$$\boxed{\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx}$$

9.1.7° $\int_{\pi/4}^{\pi/2} \frac{x\cos x}{\sin^3 x}\,dx$

$u = x$

$du = dx$

$dv = \frac{\cos x}{\sin^3 x}\,dx = \frac{d(\sin x)}{\sin^3 x} = \left| \sin x = t \right|$

$= \frac{dt}{t^3} \Big/ \circ \int$

$v = \int \frac{dt}{t^3} = \frac{t^{-3+1}}{-3+1} = \frac{-1}{2t^2} =$

$= -\frac{1}{2\sin^2 x}$

$$I = \int_{\pi/4}^{\pi/2} \frac{x\cos x}{\sin^3 x}\,dx = uv\Big|_{\pi/4}^{\pi/2} - \int_{\pi/4}^{\pi/2} u'v\,dx =$$

$$= x\cdot\left(-\frac{1}{2\sin^2 x}\right)\Big|_{\pi/4}^{\pi/2} + \frac{1}{2}\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x}$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$I = -\frac{1}{2}\left[\frac{\pi/2}{1} - \frac{\pi/4}{\left(\frac{1}{\sqrt{2}}\right)^2}\right] + \frac{1}{2}(-\operatorname{ctg} x)\Big|_{\pi/4}^{\pi/2} =$$

$$= -\frac{1}{2}\underbrace{\left[\frac{\pi}{2} - \frac{\pi}{2}\right]}_{=0} - \frac{1}{2}\left(\operatorname{ctg}\frac{\pi}{2} - \operatorname{ctg}\frac{\pi}{4}\right) =$$

$$= -\frac{1}{2}[0 - 1] = \boxed{\frac{1}{2}}$$

Zamjena promjenjive:

$$\boxed{\int_a^b f(x)\,dx = \int_\alpha^\beta f(\varphi(t))\cdot\varphi'(t)\,dt}$$

12.5*. Izračunati integral

$$I=\int_0^{\pi/3}\ln(1+\sqrt{3}\,\mathrm{tg}\,x)\,dx.$$

$$\boxed{\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx} \leftarrow \text{23. osobina}$$

$$I=\int_0^{\pi/3}\ln(1+\sqrt{3}\,\mathrm{tg}\,x)\,dx=\int_0^{\pi/3}\ln\left(1+\sqrt{3}\,\mathrm{tg}\left(\frac{\pi}{3}-x\right)\right)dx=$$

$$=\int_0^{\pi/3}\ln\left(1+\sqrt{3}\,\frac{\mathrm{tg}\frac{\pi}{3}-\mathrm{tg}\,x}{1+\mathrm{tg}\frac{\pi}{3}\mathrm{tg}\,x}\right)dx=\int_0^{\pi/3}\ln\left(1+\sqrt{3}\,\frac{\sqrt{3}-\mathrm{tg}\,x}{1+\sqrt{3}\,\mathrm{tg}\,x}\right)dx=$$

$$=\int_0^{\pi/3}\ln\left(1+\frac{3-\sqrt{3}\,\mathrm{tg}\,x}{1+\sqrt{3}\,\mathrm{tg}\,x}\right)dx=\int_0^{\pi/3}\ln\frac{1+\cancel{\sqrt{3}\,\mathrm{tg}\,x}+3-\cancel{\sqrt{3}\,\mathrm{tg}\,x}}{1+\sqrt{3}\,\mathrm{tg}\,x}\,dx$$

$$=\int_0^{\pi/3}\ln\frac{4}{1+\sqrt{3}\,\mathrm{tg}\,x}\,dx=\int_0^{\pi/3}\ln 4\,dx-\underbrace{\int_0^{\pi/3}\ln(1+\sqrt{3}\,\mathrm{tg}\,x)\,dx}_{I}=$$

$$\Rightarrow \quad I=\ln 4\cdot x\Big|_0^{\pi/3}-I$$

$$2I=\frac{\pi}{3}\ln 4$$

$$I=\frac{\pi}{6}\ln 4=\frac{\pi}{6}\ln 2^{2}=\cancel{2}\,\frac{\pi}{\cancel{6}_3}\ln 2$$

$$\boxed{I=\frac{\pi}{3}\ln 2}$$

Ušćumlić I

36fg* Pomoću određenog integrala naći graničnu vrijednost zbira

$$\lim_{n\to\infty}\underbrace{\left[\frac{1^p}{n^{p+1}}+\frac{2^p}{n^{p+1}}+\dots+\frac{n^p}{n^{p+1}}\right]}_{S},\ (p>0).$$

$$S=\frac{1}{n}\left[\left(\frac{1}{n}\right)^p+\left(\frac{2}{n}\right)^p+\dots+\left(\frac{n}{n}\right)^p\right]=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p \qquad (*)$$

Teorija: Neka je f-ja $f(x)$ def. na segmentu $[a,b]$. Podijelimo interval $[a,b]$ na n dijelova tačkama $a=x_0<x_1<x_2<\dots<x_n=b$. Uzmimo $\xi_i\in(x_{i-1},x_i)$ i posmatrajmo sumu $\sum_{i=1}^{n}f(\xi_i)\Delta x_i$, gdje je $\Delta x_i=x_i-x_{i-1}$. Ako postoji $\lim_{n\to\infty}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$ i ako je konačan, onda je

$$\boxed{\int_a^b f(x)dx=\lim_{\max\Delta x_i\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i} \qquad (**)$$

Poredeći $(*)$ i $(**)$ može se uzeti da je $\Delta x_i=\frac{1}{n}$, $\xi_i=\frac{i}{n}$ $\Rightarrow$

$$f(\xi_i)=\left(\frac{i}{n}\right)^p=\xi_i^p \Rightarrow f(x)=x^p$$

na segmentu $[0,1]$.

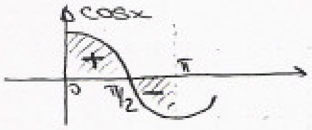
$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^p=\int_0^1 x^p dx=\frac{x^{p+1}}{p+1}\Big|_0^1=\boxed{\frac{1}{p+1}}$$

3726.* Integral $I=\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx$ nalazimo na sljedeći način:

$$\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx=\int_0^{\pi}\sqrt{\cos^2 x}\,dx=\int_0^{\pi}\cos x\,dx=\sin x\Big|_0^{\pi}=0.$$

Međutim, podintegralna f-ja $f(x)=\sqrt{\frac{1+\cos 2x}{2}}$ je pozitivna, pa vrijednost integrala mora biti veća od nule.

Gdje je greška?



$$\int_0^{\pi}\sqrt{\frac{1+\cos 2x}{2}}\,dx=\int_0^{\pi}\sqrt{\cos^2 x}\,dx=\int_0^{\pi}|\cos x|\,dx=\int_0^{\pi/2}\cos x\,dx-\int_{\pi/2}^{\pi}\cos x\,dx=$$

$$=\sin x\Big|_0^{\pi/2}-\sin x\Big|_{\pi/2}^{\pi}=\underbrace{\sin\tfrac{\pi}{2}}_{=1}-\cancelto{0}{\sin 0}-\cancelto{0}{\sin\pi}+\underbrace{\sin\pi/2}_{=1}=\boxed{2}.$$

3733.* Aproksimirati f-ju $f(x)=\int_0^{x}\frac{t}{1+t^4}\,dt$ Mac-Laurinovim polinomom drugog stepena i na osnovu toga izračunati približnu vrijednost $f\left(\frac{1}{\sqrt{10}}\right)$. Procijeniti grešku.

$$f(0)=\int_0^{0}\frac{x}{1+x^4}\,dx=0$$

$$f'(x)=\frac{d}{dx}\left[\int_0^{x}\frac{x}{1+x^4}\,dx\right]=\frac{x}{1+x^4}\Big|_0^{x}=\frac{x}{1+x^4}\Big|_{x=0}=0$$

$$f''(x)=\frac{d}{dx}\left[\frac{x}{1+x^4}\right]=\frac{1+x^4-x\cdot 4x^3}{(1+x^4)^2}=\frac{1-3x^4}{(1+x^4)^2}\Big|_{x=0}=1$$

$$\boxed{f(x)}\approx f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2=$$

$$=\boxed{\frac{x^2}{2}}$$

$$\Rightarrow f\left(\frac{1}{\sqrt{10}}\right)\approx\frac{\frac{1}{10}}{2}=\frac{1}{20}$$

Greška aproksimacije:

$$R_n(x) = \frac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1}, \quad 0<\theta<1$$

$a=0, \quad n=2$

$$R = \frac{f'''(\theta x)}{3!}x^3$$

$$f'''(x) = \left[\frac{1-3x^4}{(1+x^4)^2}\right]' = \frac{-12x^3(1+x^4)^{\cancel{2}} - (1-3x^4)\,2\cancel{(1+x^4)}\cdot 4x^3}{(1+x^4)^{\cancel{4}3}} =$$

$$= \frac{-12x^3(1+x^4) - 8x^3(1-3x^4)}{(1+x^4)^3} = \ldots = \frac{12x^7 - 20x^3}{(1+x^4)^3}$$

$$f'''(\theta x) = \frac{12\,\theta^7x^7 - 20\,\theta^3x^3}{(1+\theta^4x^4)^3}$$

$$R = \frac{12(\theta x)^7 - 20(\theta x)^3}{3!\,(1+(\theta x)^4)^3}\,x^3 \qquad x = \frac{1}{\sqrt{10}}$$

$0<\theta<1 \longrightarrow \theta=0$ nazivnik

$\theta=1$ brojnik

$$|R| < \frac{1}{3!}\cdot\left|\frac{12\cdot 1^7\cdot\left(\frac{1}{\sqrt{10}}\right)^7 - 20\cdot 1^3\cdot\left(\frac{1}{\sqrt{10}}\right)^3}{\left(1+0^4\cdot\left(\frac{1}{\sqrt{10}}\right)^4\right)^3}\cdot\left(\frac{1}{\sqrt{10}}\right)^3\right| =$$

$$= \frac{1}{3!}\cdot\left|\left[\frac{12}{10^{7/2}} - \frac{20}{10^{3/2}}\right]\cdot\frac{1}{10^{3/2}}\right| = \frac{1}{3!}\left|\frac{12}{10^5} - \frac{20}{10^3}\right| =$$

$$= \frac{1}{3!\cdot 10^3}\cdot\left|\frac{12}{10^2} - 20\right| = 0{,}0033$$

$\boxed{|R| < 0{,}0033}$ max greška aproksimacije

Nesvojstveni (nepravi) integrali

I vrste $\int_a^{+\infty} f(x)dx = \lim_{t\to+\infty} \int_a^t f(x)dx$

13.1. Izračunati integrale

5° $\int_2^{+\infty} \frac{dx}{x^2-2} = I$

$$I = \int_2^{+\infty} \frac{dx}{(x-\sqrt{2})(x+\sqrt{2})}$$

$$\frac{1}{(x-\sqrt{2})(x+\sqrt{2})} \equiv \frac{A}{x-\sqrt{2}} + \frac{B}{x+\sqrt{2}}$$

$$1 \equiv A(x+\sqrt{2}) + B(x-\sqrt{2})$$

$$1 \equiv Ax + \sqrt{2}A + Bx - \sqrt{2}B$$

$$1 \equiv \underbrace{(A+B)}_{=0}x + \underbrace{\sqrt{2}(A-B)}_{=1}$$

$A+B=0$ $\quad \sqrt{2}(A-B)=1 \Leftrightarrow A-B=\frac{1}{\sqrt{2}}$

$$A = \frac{1}{\sqrt{2}} + B$$

$$\frac{1}{\sqrt{2}} + 2B = 0$$

$$2B = -1/\sqrt{2}$$

$$\boxed{B = -\frac{1}{2\sqrt{2}}} \qquad \boxed{A} = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{2-1}{2\sqrt{2}} = \boxed{\frac{1}{2\sqrt{2}}}$$

$$I = \int_2^{+\infty} \frac{1}{2\sqrt{2}}\left(\frac{1}{x-\sqrt{2}} - \frac{1}{x+\sqrt{2}}\right)dx = \frac{1}{2\sqrt{2}}\left(\int_2^{\infty}\frac{dx}{x-\sqrt{2}} - \int_2^{\infty}\frac{dx}{x+\sqrt{2}}\right) =$$

$$= \frac{1}{2\sqrt{2}} \lim_{t\to+\infty}\left[\int_2^t \frac{dx}{x-\sqrt{2}} - \int_2^t \frac{dx}{x+\sqrt{2}}\right] =$$

$$= \frac{1}{2\sqrt{2}} \lim_{t\to+\infty}\left[\ln|x-\sqrt{2}| - \ln|x+\sqrt{2}|\right]_2^t =$$

$$= \frac{1}{2\sqrt{2}} \lim_{t\to\infty} \ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right|_2^t = \frac{1}{2\sqrt{2}} \lim_{t\to\infty}\left[\underbrace{\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|}_{\to 0} - \ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)\right]$$

$$= -\frac{1}{2\sqrt{2}}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right) = \frac{1}{2\sqrt{2}}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^{-1} = \boxed{\frac{1}{2\sqrt{2}}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}}}$$

6° $I_n = \int_0^{+\infty} x^n e^{-\lambda x} dx \quad , \lambda > 0, n \in \mathbb{N}$

$$I_n = \lim_{t \to +\infty} \int_0^t x^n e^{-\lambda x} dx = \lim_{(t)} \left[u \cdot v \Big|_0^t - \int_0^t v\, du \right]$$

$u = x^n \qquad dv = e^{-\lambda x} dx$

$du = n \cdot x^{n-1} \qquad v = -\frac{1}{\lambda} e^{-\lambda x}$

$$I_n = \lim_{t \to +\infty} \left[x^n \cdot \left(-\frac{1}{\lambda}\right) e^{-\lambda x} \Big|_0^t + \frac{n}{\lambda} \int_0^t x^{n-1} \cdot e^{-\lambda x} dx \right] =$$

$$= -\lim_{t \to +\infty} \left(x^n \cdot \frac{e^{-\lambda x}}{\lambda} \right) \Big|_0^t + \frac{n}{\lambda} \lim_{t \to +\infty} \int_0^t x^{n-1} e^{-\lambda x} dx =$$

$$= -\lim_{t \to +\infty} \frac{t^n e^{-\lambda t}}{\lambda} + \frac{n}{\lambda} \int_0^{+\infty} x^{n-1} e^{-\lambda x} dx$$

$$\lim_{t \to +\infty} \frac{t^n e^{-\lambda t}}{\lambda} \overset{\frac{\infty}{\infty}}{\underset{Lop.}{=}} \frac{1}{\lambda} \lim_{t \to \infty} \frac{n \cdot t^{n-1}}{\lambda\, e^{\lambda t}} \overset{Lop}{=} \lim_{t \to \infty} \frac{n(n-1) t^{n-2}}{\lambda^3 e^{\lambda t}} = \ldots =$$

$$= \lim_{t \to \infty} \frac{n!}{\lambda^{n+1} \underbrace{e^{\lambda t}}_{\to +\infty}} = 0$$

$$I_n = \frac{n}{\lambda} \int_0^{+\infty} x^{n-1} e^{-\lambda x} dx$$

$$I_n = \frac{n}{\lambda} I_{n-1} = \frac{n(n-1)}{\lambda^2} I_{n-2} = \ldots = \frac{n(n-1) \cdot \ldots \cdot 2 \cdot 1}{\lambda^n} I_0 = \frac{n!}{\lambda^n} I_0$$

$$I_0 = \int_0^{+\infty} e^{-\lambda x} dx = -\frac{1}{\lambda} \lim_{t \to \infty} e^{-\lambda x} \Big|_0^t = -\frac{1}{\lambda} \lim_{t \to +\infty} \left(\underbrace{e^{-\lambda t}}_{\to 0} - 1 \right) =$$

$$= \frac{1}{\lambda}$$

$$I_n = \frac{n!}{\lambda^n} \cdot \frac{1}{\lambda}$$

$$\boxed{I_n = \frac{n!}{\lambda^{n+1}}}$$

13.11*. Izračunati nesvojsvene integrale

1° $\int\limits_{-1}^{+1} \frac{dx}{x}$ $\quad x=0$ singularna tačka $\Rightarrow$ NI II vrste

$$\int\limits_{-1}^{+1} \frac{dx}{x} = \int\limits_{-1}^{0} \frac{dx}{x} + \int\limits_{0}^{1} \frac{dx}{x} = \lim_{\varepsilon\to 0+} \int\limits_{-1}^{0-\varepsilon} \frac{dx}{x} + \lim_{\eta\to 0+} \int\limits_{0+\eta}^{1} \frac{dx}{x} =$$

$$= \lim_{\varepsilon\to 0+} \ln|x|\Big|_{-1}^{-\varepsilon} + \lim_{\eta\to 0+} \ln|x|\Big|_{\eta}^{1} = \lim_{\varepsilon\to 0+} \ln\frac{\varepsilon}{1} + \lim_{\eta\to 0+} \ln\frac{1}{\eta} =$$

$$= \lim_{\varepsilon\to 0+} \underbrace{\ln\varepsilon}_{\to -\infty} - \lim_{\eta\to 0+} \underbrace{\ln\eta}_{\to -\infty} \Rightarrow \text{integral divergira}$$

2° $\int\limits_{-1}^{1} \frac{dx}{\sqrt[3]{x^2}}$ $\quad$ singularna tačka $x=0$ $\Rightarrow$ NI II vrste

$$\int\limits_{-1}^{1} \frac{dx}{\sqrt[3]{x^2}} = \int\limits_{-1}^{0} \frac{dx}{\sqrt[3]{x^2}} + \int\limits_{0}^{1} \frac{dx}{\sqrt[3]{x^2}} = \lim_{\varepsilon\to 0+} \int\limits_{-1}^{0-\varepsilon} \frac{dx}{\sqrt[3]{x^2}} + \lim_{\eta\to 0+} \int\limits_{0+\eta}^{1} \frac{dx}{\sqrt[3]{x^2}} =$$

$$= \lim_{\varepsilon\to 0+} \left(3\cdot x^{1/3}\right)\Big|_{-1}^{-\varepsilon} + \lim_{\eta\to 0+} \left(3\cdot x^{1/3}\right)\Big|_{\eta}^{1} = 3\cdot \lim_{\varepsilon\to 0+} (-\underbrace{\sqrt[3]{\varepsilon}}_{\to 0}+1) + \lim_{\eta\to 0+} 3\cdot(1-\underbrace{\sqrt[3]{\eta}}_{\to 0}) =$$

$$= 3+3 = \boxed{6}$$

3° $\int\limits_{-2}^{2} \frac{dx}{\sqrt{|1-x^2|}}$

	$-\infty$ do -1	-1 do 1	1 do $+\infty$
$1-x$	+	+	0 −
$1+x$	−	0 +	+
$1-x^2$	−	+	−

$x=\pm 1$ singularna tačka $\Rightarrow$ NI II vrste

$$\int_{-2}^{2}\frac{dx}{\sqrt{|1-x^2|}} = \int_{-2}^{-1}\frac{dx}{\sqrt{x^2-1}} + \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} + \int_{1}^{2}\frac{dx}{\sqrt{x^2-1}} =$$

$$= \lim_{\varepsilon_1\to 0_+}\int_{-2}^{-1-\varepsilon_1}\frac{dx}{\sqrt{x^2-1}} + \lim_{\substack{\varepsilon_2\to 0_+\\ \varepsilon_3\to 0_+}}\int_{-1+\varepsilon_2}^{1-\varepsilon_3}\frac{dx}{\sqrt{1-x^2}} + \lim_{\varepsilon_4\to 0_+}\int_{1+\varepsilon_4}^{2}\frac{dx}{\sqrt{x^2-1}} =$$

$$= \underbrace{\lim_{\varepsilon_1\to 0_+}\Big[\ln|x+\sqrt{x^2-1}|\Big]_{-2}^{-1-\varepsilon_1}}_{L_1} + \underbrace{\lim_{\substack{\varepsilon_2\to 0_+\\ \varepsilon_3\to 0_+}}\Big(\arcsin x\Big|_{-1+\varepsilon_2}^{1-\varepsilon_3}\Big)}_{L_2} +$$

$$+ \underbrace{\lim_{\varepsilon_4\to 0_+}\Big(\ln|x+\sqrt{x^2-1}|\Big)\Big|_{1+\varepsilon_4}^{2}}_{L_3}$$

$$L_1 = \lim_{\varepsilon_1\to 0_+}\Big[\ln|-1-\varepsilon_1+\sqrt{\cancel{1}+2\varepsilon_1+\varepsilon_1^2-\cancel{1}}| - \ln|-2+\sqrt{3}|\Big] = \ln\frac{1}{2-\sqrt{3}}$$

$$L_2 = \lim_{\substack{\varepsilon_2\to 0_+\\ \varepsilon_3\to 0_+}}\Big[\arcsin(1-\varepsilon_3) - \arcsin(-1+\varepsilon_2)\Big] = \arcsin 1 - \arcsin(-1) =$$

$$= \frac{\pi}{2}+\frac{\pi}{2} = \pi$$

$$L_3 = \lim_{\varepsilon_4\to 0}\Big[\ln|2+\sqrt{3}| - \ln|1+\varepsilon_4+\sqrt{\cancel{1}+2\varepsilon_4+\varepsilon_4^2-\cancel{1}}|\Big] = \ln(2+\sqrt{3})$$

$$I = L_1 + L_2 + L_3$$

$$I = \ln\frac{1}{2-\sqrt{3}} + \pi + \ln(2+\sqrt{3}) = \boxed{\pi + \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)}$$

14.19* Ispitati konvergenciju integrala

$$I=\int_0^{\pi/2}\ln(\sin x)\,dx$$

$x=0$ je singularna tačka $\Rightarrow$ Nesvojstveni integral II vrste

$$\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$$

$$I=\int_0^{\pi/2}\ln(\sin x)\,dx=\int_0^{\pi/2}\ln\left[\sin\left(\frac{\pi}{2}-x\right)\right]dx$$

$$\sin\left(\frac{\pi}{2}-x\right)=\underbrace{\sin\frac{\pi}{2}}_{1}\cos x-\cancel{\cos\frac{\pi}{2}}^{0}\sin x=\cos x$$

$$I=\int_0^{\pi/2}\ln\cos x\,dx$$

(+)

$$2I=\int_0^{\pi/2}\ln\sin x\,dx+\int_0^{\pi/2}\ln\cos x\,dx=\int_0^{\pi/2}\ln(\sin x\cos x)\,dx=$$

$$=\int_0^{\pi/2}\ln\left(\frac{1}{2}\sin 2x\right)dx=\int_0^{\pi/2}\ln\frac{1}{2}\,dx+\int_0^{\pi/2}\ln\sin 2x\,dx=\left|\begin{array}{c}2x=t\\ dx=dt/2\end{array}\right|=$$

$$=-\ln 2\cdot\frac{\pi}{2}+\frac{1}{2}\int_0^{\pi}\ln\sin t\,dt=\left|\begin{array}{c}t=\frac{\pi}{2}+x\\ \sin t=\cos x\\ dt=dx\end{array}\right|=$$

$$=-\frac{\pi}{2}\ln 2+\frac{1}{2}\int_{-\pi/2}^{\pi/2}\ln\cos x\,dx=-\frac{\pi}{2}\ln 2+\underbrace{\int_0^{\pi/2}\ln\cos x\,dx}_{I}$$

$$2I=-\frac{\pi}{2}\ln 2+I$$

$$\boxed{I=-\frac{\pi}{2}\ln 2}\Rightarrow (K)$$