Određeni integrali

(i nesvojstveni integrali)

1. $\int_a^b f(x)dx = \lim_{\substack{n\to\infty \\ \max \Delta x_i \to 0}} \sum_{i=1}^{n} f(x_i)\Delta x_i$

2. $\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_a^{b-\varepsilon} f(x)dx \quad , \quad \lim_{x\to b+} |f(x)| = +\infty$ nesvojstveni integral II^{e}

3. $\int_a^{\infty} f(x)dx = \lim_{b\to+\infty} \int_a^b f(x)dx \quad , \quad \lim_{x\to+\infty} f(x) = 0$ nesvojstveni integral I^{e}

4. $\int_0^1 f(x)dx = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$ [za konvergentan integral]

5. $\int_a^b f(x)dx = \int_a^b f(\xi)d\xi$ aksiom invarijantnosti

6. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

7. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ superpozicija po granicama

8. $\int_a^b c\, f(x)dx = c \int_a^b f(x)dx$

9. $\int_a^b (f_1(x) \pm f_2(x))dx = \int_a^b f_1(x)dx \pm \int_a^b f_2(x)dx$

linearnost i homogenost

10. $\int_a^b f(x)dx = F(x)\Big|_a^b = F(b-) - F(a+) \quad , \quad F'(x) = f(x)$ Lajbnicovo pravilo

11. $\int_a^b f(x)dx = \left| \begin{array}{c} x = x(t) \\ \text{neprekidna} \\ \text{bijektivna} \\ \text{invertibilna} \\ \text{diferencijabilna} \\ \text{fc-ja} \end{array} \right| = \int_{\alpha}^{\beta} f(x(t))\, x'(t)dt$ regularna smjena

12. $\int_a^b f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)dx$ parcijalna integracija

13. $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx, \quad g(x)>0, \quad a\le\xi\le b$

14. $\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx, \quad f(x)M\downarrow, \quad a\le\xi\le b$

15. $\int_a^b f(x)g(x)dx = f(b)\int_\xi^b g(x)dx, \quad f(x)M\uparrow, \quad a\le\xi\le b$

16. $\int_a^b f(x)g(x)dx = f(a)\int_a^\xi g(x)dx + f(b)\int_\xi^b g(x)dx,$ $f(x)$ monot $a\le\xi\le b$

17. $\frac{d}{dx}\int_{a(x)}^{b(x)} f(y,x)dy = f(b(x),x)\frac{db}{dx} - f(a(x),x)\frac{da}{dx} + \int_{a(x)}^{b(x)}\frac{df}{dx}dy$

$\circledast$ $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$

18. $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ izvod integrala po gornjoj granici ($f(x)$ nep.)

19. $\frac{d}{dx}\int_a^{\varphi(x)} f(t)dt = f(\varphi(x))\cdot\frac{d\varphi}{dx}$ $\circledast$ $\frac{d}{dx}\int_x^b f(t)dt = -f(x)$

20. $\int_a^b \left(\frac{d}{dx}\varphi(x)\right)dx = \varphi(b)-\varphi(a)$, $\varphi(x)$ diferencijabilna na (a,b)

21. $\int_a^b f(x)dx = \frac{b-a}{h}\int_0^h f\left(a+\frac{b-a}{h}x\right)dx, \quad (h\equiv 1)$

22. $\int_a^b f(x)dx = \frac{b-a}{d-c}\int_c^d f\left(\frac{ad-bc+(b-a)x}{d-c}\right)dx$

23. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

24. $\int_a^b f(x)dx = (b-a)\int_0^\infty f\left(\frac{a+bx}{1+x}\right)\frac{dx}{(1+x)^2}$

25. $\int_{-a}^{a} f(x)\,dx = 0 \quad , \quad f(x) = -f(-x)$

26. $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \, , \quad f(x) = f(-x)$

27. $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx = \frac{1}{2}\int_{0}^{a} [\,f(x) + f(a-x)\,]\,dx$

28. $\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} (f(x) + f(2a-x))\,dx = 0 \, , \quad f(2a-x) = -f(x)$

29. $\int_{0}^{2a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad , \quad f(2a-x) = f(x)$

30. $\int_{0}^{\infty} \frac{f(px) - f(qx)}{x}\,dx = (f(0) - f(+\infty))\ln\frac{p}{q} \quad , \ p>0, \ q>0$

$f(x)$ neprekidna na $[0,+\infty)$, $f(+\infty) = \lim\limits_{x\to+\infty} f(x) \in \mathbb{R}$

31. $\int_{0}^{\infty} \frac{f(px) - f(qx)}{x}\,dx = f(0)\ln\frac{p}{q} \quad , \ p>0, \ q>0$

$f(x)$ neprekidna, $x\in[0,\infty)$, postoji $\int_{a}^{\infty} \frac{f(x)}{x}\,dx$, $a>0$

32. $\int_{0}^{\infty} f\left(ax + \frac{b}{x}\right)dx = \frac{1}{a}\int_{0}^{\infty} f\left(\sqrt{x^2+4ab}\right)dx, \quad a>0, \ b>0$

33. $\int_{0}^{\infty} x^{k-1} f\left(ax + \frac{b}{x}\right)dx = \frac{2}{(2a)^k}\int_{2\sqrt{ab}}^{\infty} \frac{x^k f(x)}{\sqrt{x^2-4ab}}\,dx, \quad a>0, \ b>0, \ k=0, \ k=1$

34. $\int_{0}^{\infty} \frac{1}{x} f\left(\frac{x}{a} + \frac{a}{x}\right)\ln x\,dx = \ln a \cdot \int_{0}^{\infty} \frac{1}{x} f\left(\frac{x}{a} + \frac{a}{x}\right)dx, \quad a>0$

35. $\int_{0}^{\infty} \frac{1}{x} f\left(x + \frac{1}{x}\right)\operatorname{arctg} x\,dx = \frac{\pi}{4}\int_{0}^{\infty} \frac{1}{x} f\left(x + \frac{1}{x}\right)dx$

36. $\int_{-a}^{a} f(x)\frac{dx}{e^x+1} = \int_{0}^{a} f(x)\,dx \quad , \quad f(x) = f(-x)$

37. $\int_0^T f(x)dx = \int_\alpha^{\alpha+T} f(x)dx, \quad f(x) = f(x+T), \; T>0$

38. $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$

39. $\int_0^{\pi/2} f(\sin x)dx = 2\int_0^{\pi/2} f(\sin x)dx$

40. $\int_0^{\pi} x f(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx = \pi \int_0^{\pi/2} f(\sin x)dx$

41. $\frac{d}{d\lambda}\int_a^b f(x,\lambda)dx = \int_a^b \frac{df}{d\lambda}dx$

$* \int_\alpha^\beta \left(\int_a^b f(x,\lambda)dx \right) d\lambda = \int_\alpha^\beta y(\lambda)d\lambda$

→ TUTORIJAL 13 ←

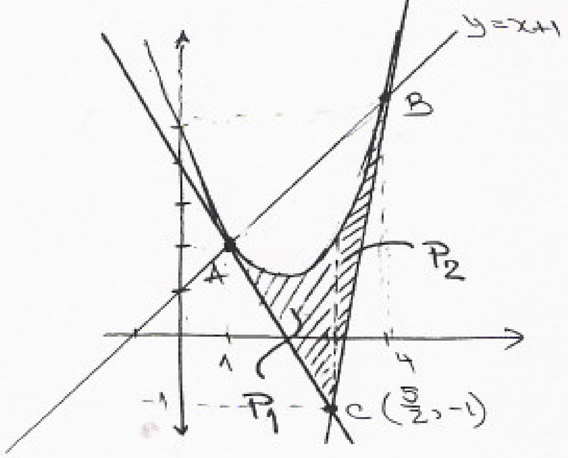
Neke primjene određenog integrala u geometriji

1.19* U presječnim tačkama prave $y=x+1$ i parabole $y=x^2-4x+5$ povučene su tangente na parabolu. Odrediti površinu lica ograničenog parabolom i tangentama.

$y = x^2-4x+5$

$D=-4<0 \Rightarrow$ nema presjeka sa x-osom

$x=0 \Rightarrow y=5$



$y=x+1$

Presječne tačke od parabole i $y=x+1$

$$\begin{aligned} y&=x+1\\ y&=x^2-4x+5 \end{aligned}$$

$x^2-4x-x+5-1=0$

$x^2-5x+4=0$

$D=25-16=9$

$x_{1,2}=\frac{5\pm3}{2} \Rightarrow x_2=4 \quad y_2=5 \quad B(4,5)$

$x_1=1 \quad y_1=2 \quad A(1,2)$

tangenta kroz A: $\quad y-2=y'(1)(x-1) \Leftrightarrow y=-2x+4$

tangenta kroz B: $\quad y-5=y'(4)(x-4) \Leftrightarrow y=4x-11$

Presjek tangenti je u tački

$C\left(\frac{5}{2},-1\right)$

$$P = P_1 + P_2 = \int_1^{5/2} (x^2-4x+5-(-2x+4))\,dx + \int_{5/2}^{4} (x^2-4x+5-(4x-11))\,dx =$$

$$= \ldots = \boxed{\frac{9}{4}}$$

12.14.e* Izračunati dužinu luka krive linije

$$\varphi = \frac{1}{2}\left(r + \frac{1}{r}\right), \quad \overset{r_1}{1} \le r \le \overset{r_2}{3}$$

Dužina luka krive je data sa

$$s = \int_{r_1}^{r_2} \sqrt{1 + r^2[\varphi'(r)]^2}\,dr.$$

$$\varphi'(r) = \frac{1}{2}\left(1 - \frac{1}{r^2}\right)$$

$$s = \int_1^3 \sqrt{1 + r^2\frac{1}{4}\left(1-\frac{1}{r^2}\right)^2}\,dr = \int_1^3 \sqrt{\frac{1}{4}\cdot 4 + \frac{r^2}{4}\left(1 - \frac{2}{r^2} + \frac{1}{r^4}\right)}\,dr =$$

$$= \frac{1}{2}\int_1^3 \sqrt{4 + r^2\left(1 - \frac{2}{r^2} + \frac{1}{r^4}\right)}\,dr = \frac{1}{2}\int_1^3 \sqrt{4 + r^2 - 2 + \frac{1}{r^2}}\,dr =$$

$$= \frac{1}{2}\int_1^3 \sqrt{2 + r^2 + \frac{1}{r^2}}\,dr = \frac{1}{2}\int_1^3 \sqrt{\frac{2r^2 + r^4 + 1}{r^2}}\,dr =$$

$$= \frac{1}{2}\int_1^3 \sqrt{\left(\frac{r^2+1}{r}\right)^2}\,dr = \frac{1}{2}\int_1^3 \frac{r^2+1}{r}\,dr = \frac{1}{2}\int_1^3 \left(r + \frac{1}{r}\right)dr =$$

$$= \frac{1}{2}\left[\frac{r^2}{2}\Big|_1^3 + \ln|r|\Big|_1^3\right] = \frac{1}{2}\left[\frac{9}{2} - \frac{1}{2} + \ln 3 - \underbrace{\ln 1}_{=0}\right] =$$

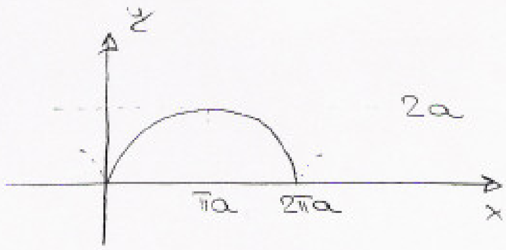
$$= \frac{1}{2}\cdot[4 + \ln 3] = \boxed{2 + \frac{1}{2}\ln 3}$$

3.9 a* Izračunati površinu obrtne površi koja nastaje rotacijom krivih datih sa:

$$x = a(t - \sin t)$$

$$y = a(1 - \cos t)$$

oko prave $y = 2a$.



cikloide

Neka luk $\widehat{AB}$ krive $f(x)$ rotira oko prave $l: Ax + By + C = 0$ i neka postoji integrabilan izvod $f'(x)$ u $[a,b]$. Ako proizvoljne normale date prave sijeku dati luk u najviše jednoj tački, onda je površina dobivene obrtne površi data sa

$$P = 2\pi \int_a^b r(f,l)\, ds = 2\pi \int_a^b r(f,l)\sqrt{1 + f'^2(x)}\, dx,$$

$$r(f,l) = \frac{|Ax + Bf(x) + C|}{\sqrt{A^2 + B^2}}.$$

$$P = 2\pi \int_a^b |y - 2a| \sqrt{1 + y'^2}\, dx = 2\pi \int_0^{2\pi} (2a - y(t))\sqrt{\dot{x}^2 + \dot{y}^2}\, dt =$$

$$= 2\pi \int_0^{2\pi} (2a - a + a\cos t)\sqrt{a^2(1 - \cos t)^2 + a^2(0 + \sin t)^2}\, dt =$$

$$= 2\pi a^2 \int_0^{2\pi} (1 + \cos t)\sqrt{1 - 2\cos t + \underbrace{\cos^2 t + \sin^2 t}_{1}}\, dt =$$

$$= 2\pi a^2 \int_0^{2\pi} (1 + \cos t)\sqrt{2(1 - \cos t)}$$

$$\sin^2 t = \frac{1 - \cos t}{2}$$

$$\cos^2 t = \frac{1 + \cos t}{2}$$

$$P = 2\pi a^2 \int_0^{2\pi} 2\cos^2 t \cdot 2 \cdot \sin t\, dt = 8\pi a^2 \int_0^{2\pi} \cos^2 t \sin t\, dt$$

$$P = \left| \begin{array}{l} \cos\frac{t}{2} = x \\ -\frac{1}{2}\sin\frac{t}{2}\frac{dt}{2} = dx \\ \sin\frac{t}{2}dt = -2dx \end{array} \right| = 8\pi a^2 \int_1^{-1} x^2 \cdot (-2)\,dx =$$

$$= -16\pi a^2 \int_1^{-1} x^2\,dx = -16\pi a^2 \frac{x^3}{3}\Big|_1^{-1} = \boxed{\frac{32}{3}\pi a^2}$$

4.1.h* Izračunati zapreminu obrtnog tijela koje nastaje obrtanjem luka ograničenog krivim datim analitički sa

$$x = \frac{y}{1+y^2}$$ (x između apscise srednje prevojne tačke i x_{max}), oko ose Oy.

$$x' = \frac{1+y^2 - y\cdot 2y}{(1+y^2)^2} = \frac{1+y^2-2y^2}{(1+y^2)^2} = \frac{1-y^2}{(1+y^2)^2}$$

$$x'' = \frac{-2y(1+y^2)^{\cancel{2}} - (1-y^2)\cdot 2\cancel{(1+y^2)}\cdot 2y}{(1+y^2)^{\cancel{4}\,3}} = \frac{2y^3-6y}{(1+y^2)^3}$$

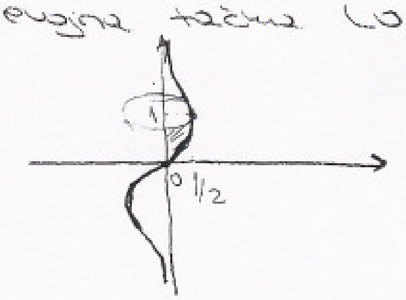
$x' = 0$ za $y = \pm 1$

	$-\infty$	-1		$+1$		$+\infty$
$1-y^2$	$-$	0	$+$	0	$-$	
y'	$\searrow$		$\nearrow$		$\searrow$	
		min		max		

$\boxed{x_{max} = x(1) = \frac{1}{2}}$

$x'' = 0$ za $y_1 = 0$, $y_{2,3} = \pm\sqrt{3}$

Srednja prevojna tačka $(0,0)$



$$V = \pi\int_0^1 x^2(y)\,dy = \pi\int_0^1 \frac{y^2}{(1+y^2)^2}\,dy$$

$$= \ldots = \frac{\pi}{8}(\pi - 2)$$