INŽENJERSKA MATEMATIKA 1

TUTORIAL 8

DIFERENCIJALNI RAČUN REALNIH FUNKCIJA JEDNE REALNE PROMJENJIVE

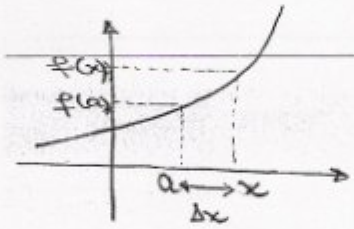
ZADACI + TEORIJA

AUTOR: BERINA HRUSTIĆ

24.11.2006. god.

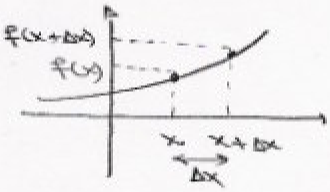
→ TUTORIJAL 8 ←

Diferencijalni račun realnih funkcija jedne realne promjenjive



$\Delta x \rightarrow$ priraštaj argumenta u tački a

$f(a+\Delta x) - f(a) \rightarrow$ priraštaj fcje u tački a



Izvodom fcje u tački a naziva se konačna granična vrijednost

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

i označava sa $f'(a)$.

Izvod fcje f u tački a se može još pisati kao:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ izvod fcje u bilo kojoj tački

Jednostrani izvodi u tački a:

$$f'_-(a) = \lim_{x \to a_-} \frac{f(x) - f(a)}{x - a} \leftarrow \text{lijevi}$$

$$f'_+(a) = \lim_{x \to a_+} \frac{f(x) - f(a)}{x - a} \leftarrow \text{desni}$$

→ Fcja ima izvod u tački a ako vrijedi

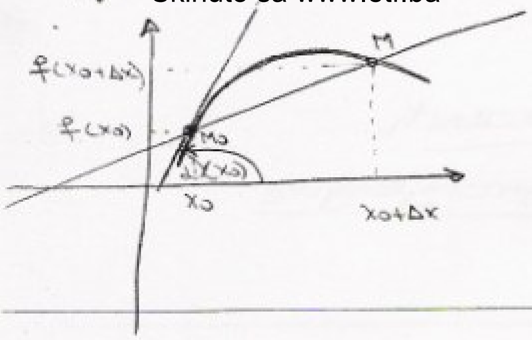
$$f'_-(a) = f'_+(a) = f'(a).$$

Beskonačan izvod

ako vrijedi

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = +\infty \text{ (ili } -\infty)$$

Geometrijsko značenje izvoda:



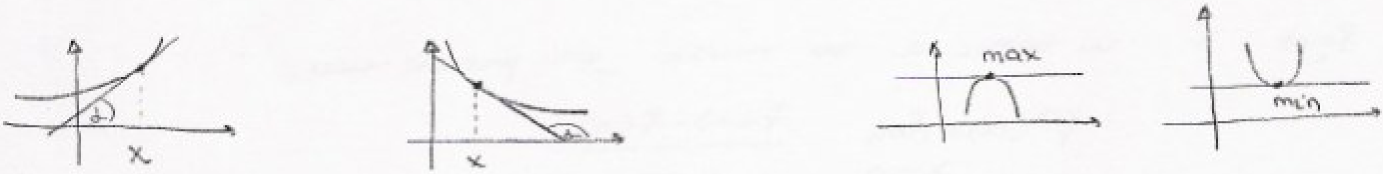
$$\boxed{\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = tg\,\alpha(x_0)}$$

tj. $f'(x_0) = tg\,\alpha(x_0)$

$f'(x_0)$ je koef. smjera tangente krive $f(x)$ povučene u tački M_0.

jednačina tangente $\quad y - f(x_0) = f'(x_0)\cdot(x-x_0)$

jednačina normale $\quad y - f(x_0) = -\dfrac{1}{f'(x_0)}(x-x_0)$



$f'(x) > 0 \Rightarrow M\uparrow$ $\qquad f'(x) < 0 \Rightarrow M\downarrow$ $\qquad f'(x)=0 \Rightarrow$ min
ekstrem f-je

Mehaničko značenje izvoda - definiranje brzine kod nejednolikog kretanja materijalne tačke.

trenutna brzina $\quad v = \dfrac{ds}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{s(t_0+\Delta t)-s(t_0)}{\Delta t}$

$a = \dfrac{dv}{dt}$

Diferencijal f-je: $\quad df(x) = f'(x_0)\cdot\Delta x$

Pravila diferenciranja:

1. $(f \pm g)'(x) = f'(x) \pm g'(x)$
2. $(f\cdot g)'(x) = f'\cdot g + f\cdot g'$
3. $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'\cdot g - f\cdot g'}{g^2}, \quad g \neq 0$

Izvod složene f-je:

$f(g(x))' = f'(g(x))\cdot g'(x)$

Izvod konstante je nula, tj:

Tablica izvoda:

funkcija	izvod
x^{α}	$\alpha \cdot x^{\alpha-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$tg\,x$	$\frac{1}{\cos^2 x}$
$ctg\,x$	$-\frac{1}{\sin^2 x}$
a^x	$a^x \ln a \Rightarrow (e^x)' = e^x$

$\log_b x$	$\frac{1}{x \ln b} \Rightarrow (\ln x)' = \frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$arc\,tg\,x$	$\frac{1}{1+x^2}$
$arc\,ctg\,x$	$-\frac{1}{1+x^2}$

$sh\,x$	$ch\,x$
$th\,x$	$\frac{1}{ch^2 x}$
$ch\,x$	$sh\,x$
$cth\,x$	$-\frac{1}{sh^2 x}$

F.M.

255.1.* Polazeći od definicije izvoda f-cije naći izvod f-cije $y=x^2$ (za bilo koji x).

$$\boxed{f'(x)} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} =$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 - \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = \boxed{2x}, \ \forall x$$

F.M.

255.3. Neka se tačka kreće po pravoj sa zakonom puta $s = 3t^3 - 2t^2 + 1$, gdje je s pređeni put mjeren u cm. U momentu t (t je vrijeme mjereno u sekundama).

1. naći srednju brzinu na razmaku $\langle t_1, t_2 \rangle$ za $t_1 = 1$ i $t_2 = 1{,}1$.
2. naći brzinu promjene date f-cije za $t = 1$.

1. $$v_{sr} = \frac{\Delta s}{\Delta t} = \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{3(t+\Delta t)^3 - 2(t+\Delta t)^2 + 1 - (3t^3 - 2t^2 + 1)}{\Delta t} =$$

$$= \frac{\cancel{3t^3} + 9t^2\Delta t + 9t\Delta t^2 + 3\Delta t^3 - \cancel{2t^2} - 4t\Delta t - 2\Delta t^2 + \cancel{1} - \cancel{3t^3} + \cancel{2t^2} - \cancel{1}}{\Delta t} =$$

$$= 9t^2 + 9t\Delta t + 3\Delta t^2 - 4t - 2\Delta t, \quad t = 1, \ \Delta t = 1{,}1 - 1 \Rightarrow \boxed{v_{sr} = 5{,}73 \ \frac{cm}{s}}$$

2. $\boxed{v} = \frac{ds}{dt} = 9t^2 - 4t = | t = 1 | = \boxed{5 \ cm/s}$

II način

$s(t_1)$ $s(t_2)$ $\Delta S = s(t_2) - s(t_1)$

$$\boxed{v_{sr} =} \frac{s(t_2)-s(t_1)}{t_2-t_1} = \frac{3t_2^3 - 2t_2^2 + 1 - 3t_1^3 + 2t_1^2 - 1}{t_2 - t_1} =$$

$$= \frac{3(t_2^3 - t_1^3) - 2(t_2^2 - t_1^2)}{t_2 - t_1} = 3(t_2^2 + t_1 t_2 + t_1^2) - 2(t_2 + t_1) =$$

$$= 3(1{,}21 + 1{,}1 + 1) - 2(1{,}1 + 1) = 9{,}93 - 4{,}2 = \boxed{5{,}73}$$

F.M.

258. Odrediti izvode sljedećih f-ja:

a) $y = \dfrac{x\sin x + \cos x}{x\cos x - \sin x}$

$$\boxed{y' =} \frac{(x\sin x + \cos x)' \cdot (x\cos x - \sin x) - (x\sin x + \cos x)(x\cos x - \sin x)'}{(x\cos x - \sin x)^2} =$$

$$= \frac{(\cancel{\sin x} + x\cdot\cos x - \cancel{\sin x})(x\cos x - \sin x) - (x\sin x + \cos x)\cdot(\cancel{\cos x} - x\sin x - \cancel{\cos x})}{(x\cos x - \sin x)^2}$$

$$= \frac{x^2\cos^2 x - \cancel{x\sin x\cos x} + x^2\sin^2 x + \cancel{x\sin x\cos x}}{(x\cos x - \sin x)^2} =$$

$$= \frac{x^2(\overbrace{\cos^2 x + \sin^2 x}^{1})}{(x\cos x - \sin x)^2} = \boxed{\frac{x^2}{(x\cos x - \sin x)^2}}$$

b) $y = \dfrac{2x}{1-x^2}$

$$\boxed{y' =} \frac{(2x)'(1-x^2) - (2x)(1-x^2)'}{(1-x^2)^2} = \frac{2(1-x^2) - 2x(0-2x)}{(1-x^2)^2} =$$

$$= \frac{2 - 2x^2 + 4x^2}{(1-x^2)^2} = \frac{2 + 2x^2}{(1-x^2)^2} = \boxed{\frac{2(1+x^2)}{(1-x^2)^2}}$$

F.M.

pravilo.

262) Naći izvode složenih f-cija $\boxed{f(g(x))' = f'(g(x))\cdot g'(x)}$

a) $y = \underbrace{\sin(\cos^2 x)}_{f_1} \cdot \underbrace{\cos(\sin^2 x)}_{f_2}$

$y' = f_1' \cdot f_2 + f_1 \cdot f_2'$

$f_1'(x) = [\sin(\cos^2 x)]' = \cos(\cos^2 x)\cdot(\cos^2 x)' = \cos(\cos^2 x)\cdot 2\cos x\cdot(-\sin x)$

$= -\underbrace{2\sin x\cos x}_{\sin 2x}\cos(\cos^2 x) = -\sin 2x\cdot\cos(\cos^2 x)$

$f_2'(x) = [\cos(\sin^2 x)]' = -\sin(\sin^2 x)\cdot(\sin^2 x)' = -\sin(\sin^2 x)\cdot\overbrace{2\sin x\cdot\cos x}^{\sin 2x} =$

$= -\sin 2x\cdot\sin(\sin^2 x)$

$\boxed{y' =} -\sin 2x\cdot\cos(\cos^2 x)\cdot\cos(\sin^2 x) - \sin(\cos^2 x)\cdot\sin 2x\cdot\sin(\sin^2 x) =$

$= -\sin 2x\,[\cos(\cos^2 x)\cos(\sin^2 x) + \sin(\cos^2 x)\sin(\sin^2 x)] =$

$= -\sin 2x\cdot\cos(\cos^2 x - \sin^2 x) = \boxed{-\sin 2x\cdot\cos(\cos 2x)}$

b) $y = x^{a^a} + a^{x^a} + a^{a^x}$

$[a^{f(x)}]' = a^{f(x)}\ln a\cdot f'(x)$

$(x^a)' = a\cdot x^{a-1}, \quad (a^x)' = a^x\ln a$

$$\boxed{y' = a^a\cdot x^{a^a-1} + a^{x^a}\cdot\ln a\cdot\overbrace{a\cdot x^{a-1}}^{(x^a)'} + a^{a^x}\cdot\ln a\cdot\overbrace{a^x\ln a}^{(a^x)'}}$$

263.b)* Dokazati da postoji (jednoznačna) f-cija $y=y(x)$ zadana sljedećom jednačinom i zatim odrediti y_x'.

$$y + \ln y = x.$$

Data jednakost definira jednoznačnu f-ciju $x=x(y)$, čiji je izvod $x_y' = 1 + \frac{1}{y}$ ($y\neq 0$). Ako je $x=x(y)$ monotona f-cija $\Rightarrow$ tada postoji jedinstvena inverzna f-cija $y=y(x)$.

Ispitivanje monotonosti po def.:

$x(y_1) - x(y_2) = (y_1-y_2) + \ln\frac{y_1}{y_2} > 0 \Leftarrow y_1 > y_2 (>0) \Rightarrow M\uparrow$

izvod inverzne f-cije

$\boxed{y_x' = \frac{1}{x_y'}} = \frac{1}{\ldots} = \boxed{\frac{y}{\ldots}}$

F.M.

266 b) Odrediti izvod sljedećih f-ja:

1° $y = \sqrt[x]{\frac{1}{x}}$

pravilo:

$$\boxed{\left[u(x)^{v(x)}\right]' = v \cdot u^{v-1} \cdot u'_x + u^v \cdot \ln u \cdot v'_x}$$

logaritamski izvod $(\ln y)'_x = \frac{y'_x}{y}$

$$y' = \left[(x^{-1})^{\frac{1}{x}}\right]' = \frac{1}{x} \cdot \left(\frac{1}{x}\right)^{\frac{1}{x}-1} \cdot (-1) \cdot x^{-2} + \left(\frac{1}{x}\right)^{\frac{1}{x}} \cdot \ln \frac{1}{x} \cdot (-1) x^{-2} =$$

$$= -\left(\frac{1}{x}\right)^{\frac{1}{x}} \frac{1}{x^2} - \frac{1}{x^2} \left(\frac{1}{x}\right)^{\frac{1}{x}} \cdot \ln \frac{1}{x} =$$

$$= \left(\frac{1}{x}\right)^{\frac{1}{x}} \left(-\frac{1}{x^2} - \frac{1}{x^2} \ln \frac{1}{x}\right) = \left(\frac{1}{x}\right)^{1/x} \frac{-1 - \ln \frac{1}{x}}{x^2} =$$

$$= \boxed{\left(\frac{1}{x}\right)^{\frac{1}{x}} \frac{\ln x - 1}{x^2}}$$

2° $y = (\sin x)^{\ln x}$ $\qquad u = \sin x, \ v = \ln x$

$$\boxed{y' =} \ln x \cdot (\sin x)^{\ln x - 1} \cdot \cos x + (\sin x)^{\ln x} \cdot \ln \sin x \cdot \frac{1}{x} =$$

$$= (\sin x)^{\ln x} \left[\ln x \cdot \frac{\cos x}{\sin x} + \frac{\ln \sin x}{x}\right] = \boxed{(\sin x)^{\ln x} \left(\frac{\ln \sin x}{x} + \ln x \cdot \mathrm{ctg}\, x\right)}$$

3° $y = \sin(x^{\ln x})$

$$y_1' = (x^{\ln x})' = \ln x \cdot x^{\ln x - 1} \cdot 1 + x^{\ln x} \cdot \ln x \cdot \frac{1}{x} = x^{\ln x}\left[\frac{\ln x}{x} + \frac{\ln x}{x}\right) = 2\frac{\ln x}{x} \cdot x^{\ln x}$$

$$y' = \left[\sin(x^{\ln x})\right]' = \cos(x^{\ln x}) \cdot (x^{\ln x})' = \boxed{\cos(x^{\ln x}) \cdot 2\frac{\ln x}{x} \cdot x^{\ln x}}$$

F.M.

270.) Za koje vrijednosti parametra α f-cija:

$$f(x)=\begin{cases} x^{\alpha}\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$$

a) neprekidna u tački $x=0$?

b) ima izvod u tački $x=0$?

c) ima neprekidan izvod u tački $x=0$?

a)

$$0\leq |x^{\alpha}\sin\tfrac{1}{x}|\leq |x^{\alpha}|\leq |x|^{\alpha}$$

$\lim\limits_{x\to 0}|x|^{\alpha}=0$ za $\alpha>0$ $\Rightarrow$ na osnovu teoreme o uklještanju imamo da je

$$\lim_{x\to 0}|x^{\alpha}\sin\tfrac{1}{x}|=0,\ \alpha>0\quad (*)$$

$\Rightarrow$ f-cija je neprekidna u $x=0$ za $\boxed{\alpha>0}$.

b)

$$f'(0)=\lim_{x\to 0}\frac{f(x+0)-f(0)}{x-0}=\lim_{x\to 0}\frac{x^{\alpha}\sin\frac{1}{x}}{x}=\lim_{x\to 0}x^{\alpha-1}\sin\frac{1}{x}=0 \text{ za}$$

na osnovu

$\alpha-1>0$

$\boxed{\alpha>1}$

c)

$$f'(x)=\alpha\cdot x^{\alpha-1}\cdot\sin\frac{1}{x}+x^{\alpha}\cdot\cos\frac{1}{x}(-1)\frac{1}{x^2}=$$

$$=\alpha\cdot x^{\alpha-1}\sin\frac{1}{x}-x^{\alpha-2}\cos\frac{1}{x}\qquad (x\neq 0)$$

Na osnovu $(*)$ je $f'(0)=0$ za $\alpha-2>0$

$\boxed{\alpha>2}$

Ušćumlić I

2069.) Naći po definiciji $f'(1)$ ako je $f(x)=\sqrt{x}$.

$$\boxed{f'(1)} = \lim_{\Delta x\to 1}\frac{f(1+\Delta x)-f(1)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\sqrt{1+\Delta x}-1}{\Delta x}\cdot\frac{\sqrt{1+\Delta x}+1}{\sqrt{1+\Delta x}+1} =$$

$$= \lim_{\Delta x\to 1}\frac{\cancel{1}+\cancel{\Delta x}-\cancel{1}}{\cancel{\Delta x}(\sqrt{1+\Delta x}+1)} = \boxed{\frac{1}{2}}$$

2072.)* Naći $f'(x)$ za $f(x)=\cos(2x+1)$: a) x u radijanima, b) x u stepenima. ~~Naći $y'=\lim$~~

a) $f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos(2x+2\Delta x+1)-\cos(2x+1)}{\Delta x}$

$$\cos x-\cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

$$\boxed{f'(x)} = \lim_{\Delta x\to 0} -\frac{2\sin\frac{4x+2\Delta x+2}{2}\sin\frac{2\Delta x}{2}}{\Delta x} =$$

$$= \lim_{\Delta x\to 0} -\frac{2\sin(2x+\Delta x+1)\sin(\Delta x)}{\Delta x} =$$

$$= \lim_{\Delta x\to 0}(-2)\sin(2x+\underbrace{\Delta x}_{\to 0}+1)\cdot\lim_{\Delta x\to 0}\underbrace{\frac{\sin\Delta x}{\Delta x}}_{\to 1} =$$

$$= \boxed{-2\sin(2x+1)}$$

b) $\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}=\frac{\pi}{180} \Rightarrow$

$$\boxed{f'(x) = -\frac{\pi}{90}\sin(2x+1)}$$

2080) Naći $f'_-(0)$ i $f'_+(0)$. Postoji li $f'(0)$ za f-ciju

$$f(x)=\begin{cases} x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

$$f'_-(0)=\lim_{\Delta x\to 0_-}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0_-}\frac{(\Delta x)^{\cancel{2}}\sin\frac{1}{\Delta x}}{\cancel{\Delta x}}=$$

$$=\lim_{\Delta x\to 0_-}\Delta x\sin\frac{1}{x}=0$$

$$f'_+(0)=\lim_{\Delta x\to 0_+}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0}\Delta x\sin\frac{1}{\Delta x}=0$$

$\Rightarrow$ $f'_-(0)=f'_+(0)\Rightarrow f'(0)$ ima smisla i

iznosi $\boxed{f'(0)=0}$

2201) Naći izvod sljedeće f-cije: $f(x)=\underbrace{3^{\frac{\sin ax}{\cos bx}}}_{f_1}+\frac{1}{3}\underbrace{\frac{\sin^3 ax}{\cos^3 bx}}_{f_2}$

$$f(x)=f_1(x)+\frac{1}{3}f_2(x)$$

$$f_1'(x)=\left(3^{\frac{\sin ax}{\cos bx}}\right)'=3^{\frac{\sin ax}{\cos bx}}\ln 3\cdot\left(\frac{\sin ax}{\cos bx}\right)'=$$

$$=3^{\frac{\sin ax}{\cos bx}}\cdot\ln 3\cdot\frac{a\cdot\cos ax\cdot\cos bx+\sin ax\cdot b\cdot\sin bx}{\cos^2 bx}$$

$$f_2'(x)=\left(\frac{\sin^3 ax}{\cos^3 bx}\right)'=\frac{3\sin^2 ax\cdot a\cdot\cos ax\cdot\cos^3 bx+\sin^3 ax\cdot 3\cos^2 bx\cdot b\sin bx}{\cos^6 bx}=$$

$$=3\cdot\frac{a\sin^2 ax\cdot\cos ax\cos^3 bx+b\cos^2 bx\sin bx\cdot\sin^3 ax}{\cos^{\cancel{6}4} bx}$$

$$\underline{\underline{f'(x)}}=f_1'(x)+\frac{1}{3}f_2'(x)=\boxed{\left[3^{\frac{\sin ax}{\cos bx}}\ln 3+\frac{\sin^2 ax}{\cos^2 bx}\right]\frac{a\cos ax\cos bx+b\sin ax\sin bx}{\cos^2 bx}}$$

Naći izvod implicitno zadanih f-ja: $\left.\begin{matrix} x=x(t) \\ y=y(t) \end{matrix}\right\} \Rightarrow y'_x = \frac{y'_t}{x'_t}$

1) $x = e^t \sin t$

$y = e^t \cos t$

$$y'_x = \frac{y'_t}{x'_t} = \frac{\cancel{e^t}\cdot \cos t - \cancel{e^t} \sin t}{\cancel{e^t} \sin t + \cancel{e^t} \cos t} = \boxed{\frac{\cos t - \sin t}{\cos t + \sin t}}$$

2) $x = \frac{1+t^3}{t^2-1}$, $y = \frac{t}{t^2-1}$

$$y'_x = \frac{y'_t}{x'_t} = \frac{\dfrac{t^2-1-t(2t)}{\cancel{(t^2-1)^2}}}{\dfrac{3t^2(t^2-1)-(1+t^3)\cdot 2t}{\cancel{(t^2-1)^2}}} = \frac{t^2-2t^2-1}{3t^4-3t^2-2t-2t^4} =$$

$$= -\frac{(t^2+1)}{t^4-3t^2-2t}.$$

3) Pokazati da $y=f(x)$ zadan parametarski $x = 2t+3t^2$ i $y = t^2+2t^3$ zadovoljava diferencijalnu jednačinu

$$2y'^3 + y'^2 - y = 0$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{2t+6t^2}{2+6t} = t$$

$$\cancel{2t^3} + \cancel{t^2} - \cancel{t^2} - \cancel{2t^3} = 0 \quad T$$

$$0 = 0$$