→ TUTORIJAL 9 ←

Taylorova formula i Taylorov polinom

Def.: Ako f-ja f u tački a ima konačan n-ti izvod, tada polinom oblika

$$T_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x$$

nazivamo Taylorovim polinomom n-tog stepena

Def.: Taylorova formula:

$$f(x) = T_n(x) + \underbrace{R_n(x)}_{\text{ostatak (greška) aproksimacije}}$$

Ostatak u Peanovom obliku: $R_n^P(x) = o((x-a)^n)$, $x \to a$

-||- u Lagrangeovom -||- : $R_n^L(x) = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a+\theta(x-a))$, $0<\theta<1$

Formula koja se dobije iz Taylorove formule za $a=0$ naziva se Maclaurinova formula.

118. Aproksimirati sljedeće f-je po Taylorovoj formuli u okolini tačke $x=0$.

a) $f(x) = e^x$

$f(0) = 1$

$f'(x) = e^x \Rightarrow f'(0) = 1$

$f'' = e^x \Rightarrow f''(0) = 1$

$\vdots$

$f^{(n)}(0) = 1$

$$f(x) = 1 + \frac{1}{1!}(x-0) + \frac{1}{2!}(x-0)^2 + \dots + \frac{1}{n!}(x-0)^n + o(x^n) =$$

$$= \boxed{1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + o(x^n)}$$

b) $f(x) = \sin x$

$f(0) = 0$

$f' = \cos x = \sin(x + \frac{\pi}{2})\big|_{x=0} = \sin\frac{\pi}{2} = 1$

$f'' = -\sin x = \sin(x + 2\frac{\pi}{2})\big|_{x=0} = \sin\pi = 0$

$f''' = -\cos x = \sin(x + 3\frac{\pi}{2})\big|_{x=0} = \sin\frac{3\pi}{2} = -1$

$f^{IV} = \sin x = \sin(x + 4\frac{\pi}{2})\big|_{x=0} = \sin 2\pi = 0$

$\vdots$

$f^{(n)} = \sin(x + n\frac{\pi}{2})$

parni izvodi jednaki nuli, a neparni su ≠ pa imamo

$f^{(2n-1)}(x) = \sin(x + (2n-1)\frac{\pi}{2})$

$= \sin(x + n\pi - \frac{\pi}{2})_{x=0}$

$= \sin(n\pi - \frac{\pi}{2}) =$

$= \cos n\pi = (-1)^{n-1}$

$$\boxed{f(x)} = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{(-1)}{3!}x^3 + \dots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1} + o(x^{2n})$$

neparni

$$= \boxed{\sum_{k=1}^{n} \frac{(-1)^{k-1}\, x^{2k-1}}{(2k-1)!} + o(x^{2n})}$$

c) $f(x) = \cos x$

$f(0) = 1$

$f' = -\sin x = \cos(x + \frac{\pi}{2})\big|_{x=0} = \cos\frac{\pi}{2} = 0$

$f'' = -\cos x = \cos(x + 2\frac{\pi}{2})\big|_{x=0} = \cos\pi = -1$

$f''' = \sin x = \cos(x + 3\frac{\pi}{2})\big|_{x=0} = \cos\frac{3\pi}{2} = 0$

$f^{IV} = \cos x = \cos(x + 4\frac{\pi}{2})\big|_{x=0} = \cos 2\pi = 1$

$\vdots$

$f^{(n)} = \cos(x + n\frac{\pi}{2})$

neparni izvodi $= 0$

$f^{(2n-1)} = \cos(x + (2n-1)\frac{\pi}{2}) = \cos(x + n\pi - \frac{\pi}{2})\big|_{x=0} = 0$

$f^{(2n)} = \cos(x + 2n\frac{\pi}{2}) = \cos(x + n\pi)\big|_{x=0} = \cos n\pi = (-1)^n$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$= \sum_{k=0}^{n} \frac{(-1)^n\, x^{2n}}{(2n)!} + o(x^{2n+1})$$

d) $f(x) = \ln(1+x)$

$f(0) = \ln 1 = 0$

$f' = \frac{1}{1+x}$

$f'' = \left(\frac{1}{1+x}\right)' = \frac{0-1}{(1+x)^2} = -(1+x)^{-2} = (-1)^{2-1} \cdot 1! \cdot (1+x)^{-2}$

$f''' = +2(1+x)^{-3} = (-1)^{3-1} \cdot 2 \cdot 1 \cdot (1+x)^{-3} = (-1)^{3-1} \cdot 2! \, (1+x)^{-3}$

$f^{IV} = -6(1+x)^{-4} = (-1)^{4-1} \cdot 3 \cdot 2 \cdot 1 \, (1+x)^{-4} = (-1)^{4-1} \cdot 3! \, (1+x)^{-4}$

$f^{V} = +24(1+x)^{-5} = (-1)^{5-1} \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot (1+x)^{-5} = (-1)^{5-1} \cdot 4! \, (1+x)^{-5}$

$\vdots$

$f^{(n)} = (-1)^{n-1}(n-1)! \, (1+x)^{-n} \Big|_{x=0} = (-1)^{n-1}(n-1)!$

$$f(x) = \cancel{f(0)}^{0} + \frac{f'}{1!}x + \frac{f''}{2!}x^2 + \dots + \frac{f^{(n)}}{n!}x^n + o(x^n) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-1} \cdot (n-1)!}{n!} \cdot x^n + o(x^n) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} \cdot x^n + o(x^n)$$

e) $f(x) = \ln(1-x)$

Rezultat dobijemo ako umjesto x u zad d) stavimo $-x$, tj. $x = -x$:

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} \cdot (-x)^n + o(x^n) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} \cdot (-1)^n \cdot x^n + o(x^n) =$$

neparno, pa uvijek imamo (-1)

$$= \sum_{k=1}^{n} \frac{(-1)^{2n-1}}{n} \cdot x^n + o(x^n) =$$

$$= -\sum \frac{x^n}{n} + o(x^n)$$

d) $f(x) = \ln(1+x)$

$f(0) = \ln 1 = 0$

$f' = \frac{1}{1+x}$

$f'' = \left(\frac{1}{1+x}\right)' = \frac{0-1}{(1+x)^2} = -(1+x)^{-2} = (-1)^{2-1} 1! \cdot (1+x)^{-2}$

$f''' = +2(1+x)^{-3} = (-1)^{3-1} \cdot 2 \cdot 1 \cdot (1+x)^{-3} = (-1)^{3-1} \cdot 2! \, (1+x)^{-3}$

$f^{IV} = -6(1+x)^{-4} = (-1)^{4-1} \cdot 3 \cdot 2 \cdot 1 \, (1+x)^{-4} = (-1)^{4-1} \cdot 3! \, (1+x)^{-4}$

$f^{V} = +24(1+x)^{-5} = (-1)^{5-1} \cdot 4 \cdot 3 \cdot 2 \cdot 1 \cdot (1+x)^{-5} = (-1)^{5-1} \cdot 4! \, (1+x)^{-5}$

$\vdots$

$f^{(n)} = (-1)^{n-1} (n-1)! \, (1+x)^{-n} \Big|_{x=0} = (-1)^{n-1} (n-1)!$

$f(x) = \cancel{f(0)}^{0} + \frac{f'}{1!} x + \frac{f''}{2!} x^2 + \dots + \frac{f^{(n)}}{n!} x^n + o(x^n) =$

$= \sum_{k=1}^{n} \frac{(-1)^{n-1} \cdot (n-1)!}{n!} \cdot x^n + o(x^n) =$

$= \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} \cdot x^n + o(x^n)$

e) $f(x) = \ln(1-x)$

Rezultat dobijemo ako umjesto x u zad d) stavimo $-x$, tj. $x = -x$:

$f(x) = \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} \cdot (-x)^n + o(x^n) =$

$= \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n} \cdot (-1)^n \cdot x^n + o(x^n) =$

neparno, pa uvijek imamo (-1)

$= \sum_{k=1}^{n} \frac{(-1)^{2n-1}}{n} \cdot x^n + o(x^n) =$

$= -\sum \frac{x^n}{n} + o(x^n)$

429* Napisati razlaganje sljedećih funkcija po potencijama od x.

a) $f(x) = e^{\sin x}$ do x^3.

$$f' = (e^{\sin x})' = e^{\sin x}\cdot(\sin x)' = e^{\sin x}\cos x \,\Big|_{x=0} = e^0\cdot 1 = 1$$

$$f'' = e^{\sin x}\cdot\cos x\cdot\cos x - e^{\sin x}\cos x\sin x \,\Big|_{x=0} =$$

$$f(0) = e^0 = 1 \qquad = e^0\cdot 1 - e^0\cdot 0 = 1 - 0 = 1$$

$$\boxed{e^{\sin x}} = f(0) + \frac{f'}{1!}\cdot x + \frac{f''}{2!}x^2 + o(x^3) =$$

$$= \boxed{1 + \frac{x}{1} + \frac{x^2}{2} + o(x^3)} \leftarrow \text{u okolini tačke } x=0$$

b) $f(x) = \ln\cos x$ do x^6

U okolini tačke $x=0$ je:

$$f(0) = \ln 1 = 0$$

$$f' = \frac{1}{\cos x}\cdot(-\sin x) = -\frac{\sin x}{\cos x} = -\operatorname{tg} x \,\Big|_{x=0} = 0$$

$$f'' = -\frac{1}{\cos^2 x}\Big|_{x=0} = -1$$

$$f''' = -\frac{0 + 2\cos x\sin x}{\cos^4 x} = -\frac{2\sin x}{\cos^3 x}\Big|_{x=0} = 0$$

$$f^{(IV)} = -2\cdot\frac{\cos x\cdot\cos^3 x + \sin x\cdot 3\cos^2 x\cdot\sin x}{\cos^6 x} = -2\cdot\frac{\cos^4 x + 3\cos^2 x\sin^2 x}{\cos^6 x} =$$

$$= -2\,\frac{\cos^2 x + 3\sin^2 x}{\cos^4 x}\Big|_{x=0} = -2$$

$$f^{(V)} = -2\cdot\frac{(-2\cos x\sin x + 6\sin x\cos x)\cdot\cos^4 x + (\cos^2 x + 3\sin^2 x)\cdot 4\cos^3 x\sin x}{\cos^8 x}$$

$$= -2\,\frac{4\cos^5 x\sin x + 4\cos^5 x\sin x + 12\cos^3 x\sin^3 x}{\cos^8 x}$$

$$= -2\,\frac{8\cos^5 x\sin x + 12\cos^3 x\sin^3 x}{\cos^8 x} = -2\,\frac{8\cdot\cos^2 x\sin x + 12\sin^3 x}{\cos^5 x}\Big|_{x=0} = 0$$

$$f^{(VI)} = -2\left(\frac{8\cos^3 x \sin x + 12\sin^3 x}{\cos^5 x}\right)' =$$

$$= -2\,\frac{\left[8(-2\cos x\sin^2 x + \cos^3 x) + 36\sin^2 x\cos x\right]\cos^5 x - (8\cos^3 x\sin x + 12\sin^3 x)([illegible]\sin x)}{\cos^{10} x}\Bigg|_{x=0}$$

$$= -2\cdot 8 = -16$$

$$\ln\cos x = -\frac{x^2}{2} - \frac{2}{4\cdot 3\cdot 2}x^4 - \frac{16}{6\cdot 5\cdot 4\cdot 3\cdot 2}x^6 + o(x^6) =$$

$$= \boxed{-\frac{x^2}{2} - \frac{x^2}{12} - \frac{x^6}{45} + o(x^6)}$$

Ušćumlić I

2691. Aproksimirati funkciju $y=x^2\ln^2x$ u okolini tačke $x=1$ Taylorovim polinomom četvrtog stepena i procijenuti grešku aproksimacije za $x\in\left[\frac{9}{10},\frac{11}{10}\right]$.

$f(x)=x^2\ln^2x$

$f(1)=1^2\cdot\ln^2 1=0$

$f'=(x^2\ln^2x)'=2x\cdot\ln^2x+x^2\cdot 2\ln x\cdot\frac{1}{x}=2x\ln^2x+2x\ln x=$

$=2x\ln x(\ln x+1)\big|_{x=1}=2\cdot 1\cdot 0(0+1)=0$

$f''=2\left[x\ln x(\ln x+1)\right]'=2\cdot\left[\left(1\cdot\ln x+x\cdot\frac{1}{x}\right)(\ln x+1)+(x\ln x)\cdot\frac{1}{x}\right]=$

$=2\left[(\ln x+1)^2+\ln x\right]\big|_{x=1}=2\cdot[0+1+0]=2$

$f'''=2\left[\ln^2x+3\ln x+1\right]'=2\left[2\ln x\cdot\frac{1}{x}+\frac{3}{x}\right]=\frac{4}{x}\ln x+\frac{6}{x}=\frac{1}{x}(4\ln x+6)\big|_{x=1}=$

$=1\cdot(4\cdot 0+6)=6$

$f^{(IV)}=\left[\frac{1}{x}(4\ln x+6)\right]'=\left(-\frac{1}{x^2}\right)(4\ln x+6)+\frac{1}{x}\cdot 4\cdot\frac{1}{x}=-\frac{4}{x^2}\ln x-\frac{6}{x^2}+\frac{4}{x^2}=$

$=-\frac{4}{x^2}\ln x-\frac{2}{x^2}=-\frac{1}{x^2}(4\ln x+2)\big|_{x=1}=(-1)(4\cdot 0+2)=-2$

$x^2\ln^2x=0+0+\frac{2}{2!}(x-1)^2+\frac{6}{3!}(x-1)^3-\frac{2}{4!}(x-1)^4+R_4(x)=$

$$=\boxed{(x-1)^2+(x-1)^3-\frac{1}{12}(x-1)^4+R_4(x)}$$

Procjena greške: $R_n(x)=\dfrac{f^{(n+1)}(a+\theta(x-a))}{(n+1)!}(x-a)^{n+1},\ 0<\theta<1$

$R_4(x)=\dfrac{f^{(V)}(1+\theta(x-1))}{5!}(x-1)^5$

$f^{(V)}(x)=-\left[\frac{1}{x^2}(4\ln x+2)\right]'=-\left[-2\cdot\frac{1}{x^3}(4\ln x+2)+\frac{1}{x^2}\left(\frac{4}{x}\right)\right]=$

$=-\left[\frac{1}{x^3}\cdot(\cancel{4}-8\ln x-\cancel{4})\right]=\frac{8\ln x}{x^3}$

$$f^{(v)}(1+\theta(x-1)) = \frac{8\ln(1+\theta(x-1))}{[1+\theta(x-1)]^3}$$

$$R_4(x) = \frac{8(x-1)^5}{5!} \cdot \frac{\ln[1+\theta(x-1)]}{[1+\theta(x-1)]^3}$$

Hoćemo da vidimo koja je max greška aproksimacije

$$|R_4| = \frac{8}{5!}\left|(x-1)^5 \frac{\ln[1+\theta(x-1)]}{[1+\theta(x-1)]^3}\right| \quad (*)$$

$x \in \left(\frac{9}{10}, \frac{11}{10}\right) \Rightarrow (*)$ će imati max vrijednost za $x = \frac{11}{10}$

$0 < \theta < 1 \Rightarrow$ nazivnik $\theta = 0$

brojnik $\theta = 1$

$$\Rightarrow |R_4| < \frac{8}{5!} \cdot \big(\overbrace{\tfrac{11}{10} - 1}^{1/10}\big)^5 \cdot \frac{\ln\left[1 + 1 \cdot \frac{1}{10}\right]}{1+0} =$$

$$= \frac{8}{5!} \cdot \frac{1}{10^5} \cdot \frac{\ln\left(1+\frac{1}{10}\right)}{1} =$$

$$= \frac{8}{5!} \cdot \frac{1}{10^5} \cdot \ln\left(1+\frac{1}{10}\right) = \frac{1}{15} \cdot \frac{1}{10^5} \ln\left(1+\frac{1}{10}\right)$$

$$\ln(1+x) \approx x, \quad x \to 0$$

$$\Rightarrow |R_4| < \frac{1}{15} \cdot \frac{1}{10^5} \cdot \frac{1}{10} = \frac{1}{15 \cdot 10^6}$$

$$\boxed{|R_4| < 6{,}6 \cdot 10^{-8}}$$

max
greška
aproksimacije

2712.* Odrediti konstante a i b tako da je

$$\cos x - a\sin x + b\ln(1+x) - 1 = o(x^3),\quad x\to 0$$

Na osnovu zadatka 418. b), c), d.) imamo da je:

$$\left.\begin{aligned}
\cos x &= \sum_{k=0}^{n} \frac{(-1)^n x^{2n}}{(2n)!} + o(x^{2n+1})\\
\sin x &= \sum_{k=1}^{n} \frac{(-1)^{k-1} x^{2n-1}}{(2n-1)!} + o(x^{2n})\\
\ln(1+x) &= \sum_{k=1}^{n} \frac{(-1)^{n-1}}{n}\cdot x^n + o(x^n)
\end{aligned}\right\}\quad x\to 0$$

$$\cos x \approx 1 - \frac{x^2}{2!}$$

$$\sin x \approx x - \frac{1}{3!}x^3$$

$$\ln(1+x) \approx x - \frac{1}{2!}x^2 + \frac{2}{3!}x^3$$

$$\cos x - a\sin x + b\ln(x+1) - \cancel{1} = \cancel{1} - \frac{x^2}{2} - a\left(x - \frac{x^3}{6}\right) + b\left(x - \frac{x^2}{2} + \frac{1}{3}\cdot x^3\right) + o$$

$$= -\frac{x^2}{2} - ax + \frac{ax^3}{6} + bx - \frac{bx^2}{2} + \frac{b}{3}x^3 + o(x^3) =$$

$$= \underbrace{-(1+b)}_{=0}\frac{x^2}{2} + x\underbrace{(b-a)}_{=0} + \frac{x^3}{3}\left(\frac{a}{2}+b\right) + o(x^3)$$

$$\Rightarrow \quad 1+b=0 \Rightarrow b=-1$$

$$b-a=0 \Rightarrow a=-1$$

Dakle, $\boxed{a=b=-1}$