by Sejla Cebiric

ZADACI - Var. A :
za drugi parcijalni ispit iz IM1, 08. 01. 2007.

Zad. 1. Izračunajte (ili ustanovite da ne postoji) sljedeći limes funkcije koristeći asimptotsku relaciju $a^x = 1 + x\ln a + o(x)$ $(x \to 0), (a > 0)$:

$$\lim_{x\to 0}\left(\frac{1+4^x+5^x}{3}\right)^{\frac{1}{x}}$$

I. 1.
II. $\sqrt[3]{20}$.
III. 10.
IV. Dati limes ne postoji.

Zad.2. Izračunajte derivaciju funkcije

$$f(x) := \int_{-x}^{x}\frac{t^2+1}{t-1}dt$$

u tački $x = \frac{1}{2}$.

I. $f'\left(\frac{1}{2}\right) = -\frac{10}{3}$. II. $f'\left(\frac{1}{2}\right) = \frac{10}{3}$.

III. $f'\left(\frac{1}{2}\right) = \frac{2}{3}$. IV. $f'\left(\frac{1}{2}\right) = -\frac{2}{3}$.

Zad. 3. Izračunajte površinu lika kojeg ograničavaju: grafik funkcije φ zadane formulom

$$\varphi(x) = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}(1 + x^2\mathrm{tg}^2 t)^{-1}dt ,$$

x – osa i ordinate u tačkama čije su apscise 0 i x.

I. $\ln(1 + 2x)$.
II. $\ln(1 + |x|)$.
III. $\frac{1}{1+|x|}$.
IV. $\frac{1}{1+2x}$.

Zad. 4. Zadana je funkcija f formulom

$$f(x)=\sum_{n=1}^{\infty} nx(1+n^5x^2)^{-1}.$$

Ispitajte integrabilnost zadane funkcije f, a zatim izračunajte integral $\int_0^x f(t)dt$.

I. $\sum_{n=1}^{\infty}\frac{1}{2n^{\frac{3}{2}}}\ln(1+n^5x^2)$. II. $\sum_{n=1}^{\infty}\frac{1}{2n^6}\ln(1+n^5x^2)$.

III. $\sum_{n=1}^{\infty}\frac{1}{2n^4}\ln(1+n^4x)$. IV. $\sum_{n=1}^{\infty}\frac{1}{2n^4}\ln(1+n^5x^2)$.

Zad. 5. Realna funkcija f jedne realne promjenljive zadana je formulom

$$f(x):=\operatorname{sgn}(x)\cdot\ln(x-\sqrt{n^2-x^2}),$$

gdje je n najmanja cifra Vašeg jedinstvenog matičnog broja koja je veća od 0.

a) Odredite prirodni domen Dom (f) i ispitajte ponašanje funkcije f na rubovima područja Dom (f).
b) Odredite eventualne presjeke grafika G(f) sa koordinatnim osama i ispitajte znak zadane funkcije f.
c) Odredite eventualne tačke prekida i singulariteta i klasificirajte ih za zadanu funkciju f i njenu recipročnu funkciju $\frac{1}{f}$.
d) Odredite intervale monotonosti i eventualne tačke lokalnog i apsolutnog ekstrema zadane funkcije f, kao i eventualne prelomne i povratne tačke grafika njene recipročne funkcije $\frac{1}{f}$.
e) Ispitajte konveksnost i konkavnost i odredite eventualne prevojne tačke zadane funkcije f.
f) Odredite sliku Im(f) i nacrtajte grafik zadane funkcije f.

Rješenja zadataka iz IM1 (Var. A)

(Z1) $a^x = 1 + x\ln a + o(x) \quad (x \to 0), (a>0)$

$$\lim_{x\to 0}\left(\frac{1+4^x+5^x}{3}\right)^{\frac{1}{x}}$$

Rješenje:

$$L = \lim_{x\to 0}\left(\frac{1+4^x+5^x}{3}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(\frac{1+1+x\ln 4+1+x\ln 5}{3}\right)^{\frac{1}{x}} =$$

$$= \lim_{x\to 0}\left(1+x\cdot\ln\sqrt[3]{20}\right)^{\frac{1}{x}} = \lim_{x\to 0}\Bigg[\underbrace{\left(1+x\ln\sqrt[3]{20}\right)^{\frac{1}{x\ln\sqrt[3]{20}}}}_{\to e}\Bigg]^{\ln\sqrt[3]{20}} =$$

$$= e^{\ln\sqrt[3]{20}} = \sqrt[3]{20}$$

$$\boxed{L = \sqrt[3]{20}}$$

(Z2) $f(x) = \int_{-x}^{x} \frac{t^2+1}{t-1}\,dt, \quad f'(\frac{1}{2}) = ?$

Rješenje:

1. način

$$f(x) = \int_{\varphi(x)}^{\psi(x)} F(t)\,dt \Rightarrow f'(x) = F(\psi(x))\psi'(x) - F(\varphi(x))\cdot\varphi'(x)$$

$$f'(x) = \frac{x^2+1}{x-1} + \frac{(-x)^2+1}{-x-1} = \frac{x^2+1}{x-1} - \frac{x^2+1}{x+1} = \frac{(x^2+1)\cdot 2}{x^2-1}$$

$$f'(\tfrac{1}{2}) = \frac{\frac{5}{4}\cdot 2}{-\frac{3}{4}} = -\frac{10}{3}$$

$$\boxed{f'(\tfrac{1}{2}) = -\frac{10}{3}}$$

2. način

$$f(x)=\underbrace{\int_{-x}^{0}\frac{t^2+1}{t-1}\,dt}_{I_1}+\underbrace{\int_{0}^{x}\frac{t^2+1}{t-1}\,dt}_{I_2}$$

$$I_1=\int_{-x}^{0}\frac{t^2+1}{t-1}\,dt=\left|\begin{array}{l} t=-y \\ dt=-dy \\ t=-x\to y=x \\ t=0\to y=0\end{array}\right|=\int_{x}^{0}\frac{y^2+1}{-y-1}(-dy)\overset{y=t}{=}-\int_{0}^{x}\frac{t^2+1}{t+1}\,dt$$

$$f(x)=-\int_{0}^{x}\frac{t^2+1}{t+1}\,dt+\int_{0}^{x}\frac{t^2+1}{t-1}\,dt$$

$$f'(x)=-\frac{x^2+1}{x+1}+\frac{x^2+1}{x-1}=\frac{(x^2+1)\cdot 2}{x^2-1}$$

$$f'\left(\frac{1}{2}\right)=\frac{\frac{5}{4}\cdot 2}{-\frac{3}{4}}=-\frac{10}{3}$$

$$\boxed{f'\left(\frac{1}{2}\right)=-\frac{10}{3}}$$

(23) $\varphi(x)=\dfrac{2}{\pi}\displaystyle\int_{0}^{\frac{\pi}{2}}(1+x^2\,tg^2t)^{-1}\,dt$

Rješenje:

$$\varphi(x)=\frac{2}{\pi}\int_{0}^{\frac{\pi}{2}}\frac{dt}{1+x^2tg^2t}-\left|\begin{array}{l} tg\,t=y \\ t=arc\,tg\,y \\ dt=\dfrac{dy}{1+y^2}\end{array}\right|=\frac{2}{\pi}\int_{0}^{\infty}\frac{\frac{dy}{1+y^2}}{1+x^2y^2}=$$

$$=\frac{2}{\pi}\int_{0}^{\infty}\frac{dy}{(1+y^2)(1+x^2y^2)}$$

by Sejla Cebiric

$$\frac{1}{(1+y^2)(1+x^2y^2)} \equiv \frac{Ay+B}{1+y^2} + \frac{Cy+D}{1+x^2y^2}$$

$$1 \equiv Ay + B + Ax^2y^3 + Bx^2y^2 + Cy + D + Cy^3 + Dy^2$$

$$\left.\begin{array}{l} Ax^2 + C = 0 \\ Bx^2 + D = 0 \\ A + C = 0 \end{array}\right\} \Rightarrow A = C = 0$$

$$B + D = 1 \Rightarrow D = 1 - B$$

$$Bx^2 + 1 - B = 0 \Rightarrow \quad B = \frac{-1}{x^2-1} \quad , \ x \neq \pm 1$$

$$D = \frac{x^2}{x^2-1}$$

$$\varphi(x) = \frac{2}{\pi} \cdot \frac{-1}{x^2-1} \int_0^\infty \frac{dy}{1+y^2} + \frac{2}{\pi} \cdot \frac{x^2}{x^2-1} \int_0^\infty \frac{dy}{1+x^2y^2} =$$

$$= \frac{2}{\pi} \cdot \frac{-1}{x^2-1} \cdot \operatorname{arctg} y \Big|_0^\infty + \frac{2}{\pi} \frac{x^2}{x^2-1} \cdot \frac{1}{|x|} \operatorname{arctg}(|x|y) \Big|_0^\infty =$$

$$= \frac{\cancel{2}}{\cancel{\pi}} \cdot \frac{-1}{x^2-1} \cdot \frac{\cancel{\pi}}{\cancel{2}} + \frac{\cancel{2}}{\cancel{\pi}} \cdot \frac{|x|}{x^2-1} \cdot \frac{\cancel{\pi}}{\cancel{2}} = \frac{|x|-1}{x^2-1} = \frac{1}{|x|+1}$$

$$\int_0^x \varphi(\xi)\, d\xi = \int_0^x \frac{d\xi}{|\xi|+1} = \ln(|\xi|+1) \Big|_0^x = \ln(|x|+1)$$

by Sejla Cebiric

54) $f(x)=\sum_{n=1}^{\infty}\frac{nx}{1+n^5x^2}$

Rjesenje:

$\varphi(x)=\frac{nx}{1+n^5x^2}$ def.-nep. $\forall x\in\mathbb{R}$

$$\varphi'(x)=\frac{n+n^6x^2-2n^6x^2}{(1+n^5x^2)^2}=\frac{n(1-n^5x^2)}{(1+n^5x^2)^2}=0$$

$$1-n^5x^2=0 \Rightarrow x_{1,2}=\pm\frac{\sqrt{n}}{n^3}$$

$$\varphi''(x)=n\cdot\frac{-2n^5x(1+n^5x^2)^{\cancel{2}}-(1-n^5x^2)\cdot 2\cancel{(1+n^5x^2)}\cdot 2n^5x}{(1+n^5x^2)^{\cancel{4}\,3}}$$

$$\varphi''(x)=\frac{-2n^6x(1+n^5x^2+2-2n^5x^2)}{(1+n^5x^2)^4}=\frac{-2n^6x(3-n^5x^2)}{(1+n^5x^2)^3}$$

$\varphi''\left(\frac{\sqrt{n}}{n^3}\right)=\frac{-n^3\sqrt{n}}{4}<0 \Rightarrow x_1=\frac{\sqrt{n}}{n^3}$ MAX

$\varphi''\left(-\frac{\sqrt{n}}{n^3}\right)=\frac{n^3\sqrt{n}}{4}>0 \Rightarrow x_2=-\frac{\sqrt{n}}{n^3}$ MIN

$\Rightarrow |\varphi(x)|\leq\varphi(x_1) \Rightarrow \left|\frac{nx}{1+n^5x^2}\right|\leq\frac{1}{2n^{\frac{3}{2}}}$ $\alpha=\frac{3}{2}>1 \Rightarrow$

$\Rightarrow$ red $\sum_1^{\infty}\frac{1}{n^{\frac{3}{2}}}$ je (k) $\Rightarrow$ red $\sum_1^{\infty}\varphi(x)$ je UNIF. (k) pa je $f(x)$ integrabilna i možemo je integraliti član po član.

$$\int_0^x f(t)\,dt=\int_0^x\sum_{n=1}^{\infty}\frac{nt}{1+n^5t^2}\,dt=\sum_{n=1}^{\infty}\int_0^x\frac{n\,t\,dt}{1+n^5t^2}=\left|\begin{matrix}n^5t^2=z\\ t\,dt=\frac{dz}{2n^5}\end{matrix}\right|=$$

$$=\sum_{n=1}^{\infty}\frac{1}{2n^4}\int_0^{n^5x^2}\frac{dz}{1+z}=\sum_{n=1}^{\infty}\frac{1}{2n^4}\cdot\ln(1+n^5x^2)$$

5) $f(x) = \operatorname{sgn}(x) \cdot \ln\left(x - \sqrt{m^2 - x^2}\right)$

Rješenje:

$$m = 1 \quad \Rightarrow \quad f(x) = \operatorname{sgn}(x) \cdot \ln\left(x - \sqrt{1 - x^2}\right)$$

a) $1 - x^2 \geq 0 \quad \wedge \quad x - \sqrt{1 - x^2} > 0$

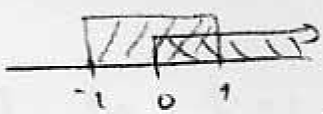
$x^2 \leq 1$

$|x| \leq 1$

$x \in [-1, 1]$

$\sqrt{1 - x^2} < x$

$1 - x^2 \geq 0 \qquad x > 0$



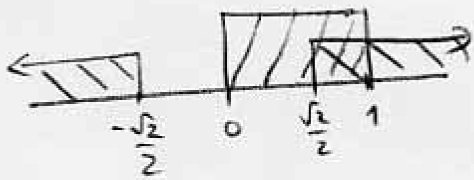
$x \in (0, 1]$

$\sqrt{1 - x^2} < x \; /^2$

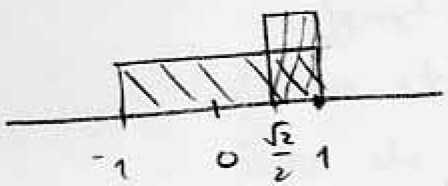
$1 - x^2 < x^2$

$x^2 > \frac{1}{2}$

$|x| > \frac{\sqrt{2}}{2}$



$x \in \left(\frac{\sqrt{2}}{2}, 1\right]$



$\text{Dom}(f): \; x \in \left(\frac{\sqrt{2}}{2}, 1\right]$

Kako je $f(x)$ definisana samo za vrijednosti varijable x koje su pozitivne to je $\operatorname{sgn}(x) = 1$, pa ćemo u nastavku zadatka za $f(x)$ pisati:

$$f(x) = \ln\left(x - \sqrt{1 - x^2}\right).$$

$$\lim_{x \to \frac{\sqrt{2}}{2}+} f(x) = \lim_{x \to \frac{\sqrt{2}}{2}+} \ln\left(\underbrace{x - \sqrt{1 - x^2}}_{\to 0_+}\right) = -\infty$$

Možemo zaključiti da u tački $x = \frac{\sqrt{2}}{2}+$ imamo vertikalnu asimptotu.

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} \ln(x - \sqrt{1-x^2}) = 0$$

$$f(1) = 0$$

Vidimo da je funkcija $f(x)$ u tački $x=1$ neprekidna s lijeva.

b) Kako je $x \in (\frac{\sqrt{2}}{2}, 1]$ to $f(x)$ neće imati presječnih tačaka sa Oy osom.

$$y=0 \iff \ln(x-\sqrt{1-x^2}) = 0$$

$$x - \sqrt{1-x^2} = 1$$

$$\sqrt{1-x^2} = x-1$$

desna strana jednačine je uvijek ≤ 0 tako da je jedino rješenje jednačine $x=1$.

Presjek $f(x)$ i Ox ose je tačka $A(1,0)$.

ZNAK:

$$y>0 \iff \ln(x-\sqrt{1-x^2}) > 0$$

$$x-\sqrt{1-x^2} > 1$$

$\sqrt{1-x^2} < x-1$ kako je $x-1 \leq 0$ $\forall x \in DP$. to je rješenje nejdnačine $x \in \emptyset$

$$y<0 \iff \ln(x-\sqrt{1-x^2}) < 0$$

$$x-\sqrt{1-x^2} < 1$$

$\sqrt{1-x^2} > x-1$ koko je $x-1 \leq 0$ $\forall x \in DP$ to je rješenje nejednačine $x \in (\frac{\sqrt{2}}{2}, 1)$

) F-ija $f(x)$ je neprekidna $\forall x \in Dom(f)$.

$$g(x) = \frac{1}{f(x)} = \frac{1}{\ln(x-\sqrt{1-x^2})}$$

$g(x)$ je definisana za $x \in \left(\frac{\sqrt{2}}{2}, 1\right)$, a i neprekidna $\forall x \in \left(\frac{\sqrt{2}}{2}, 1\right)$

d) $f(x) = \ln(x - \sqrt{1-x^2})$

$$f'(x) = \frac{1}{x-\sqrt{1-x^2}} \cdot \left(1 + \frac{x}{\sqrt{1-x^2}}\right) = \frac{x+\sqrt{1-x^2}}{(x-\sqrt{1-x^2})\sqrt{1-x^2}}, \quad x \in \left(\frac{\sqrt{2}}{2}, 1\right)$$

$$\lim_{x \to 1_-} f'(x) = \lim_{x \to 1_-} \frac{x+\sqrt{1-x^2}}{(x-\sqrt{1-x^2})\sqrt{1-x^2}} = +\infty = \operatorname{tg} \alpha_{vt} \Rightarrow \alpha_{vt} = \frac{\pi}{2}$$
ili $\alpha_{vt} = -\frac{\pi}{2}$

$$f'(x) > 0 \Rightarrow \frac{x+\sqrt{1-x^2}}{(x-\sqrt{1-x^2})\sqrt{1-x^2}} > 0 \quad \text{T} \quad \forall x \in \left(\frac{\sqrt{2}}{2}, 1\right)$$

$\Rightarrow f(x) \nearrow \forall x \in \left(\frac{\sqrt{2}}{2}, 1\right)$

$$g(x) = \frac{1}{f(x)}$$

$$g'(x) = \frac{1}{\ln^2(x-\sqrt{1-x^2})} \cdot f'(x)$$

$g(x)$ nema prelomnih i povratnih tačaka.

$$f'(x) = \frac{x + \sqrt{1-x^2}}{(x-\sqrt{1-x^2})\sqrt{1-x^2}}$$

$$f''(x) = \frac{\left(1 - \frac{x}{\sqrt{1-x^2}}\right)(x-\sqrt{1-x^2})\sqrt{1-x^2} - (x+\sqrt{1-x^2})\cdot\left[(x-\sqrt{1-x^2})\sqrt{1-x^2}\right]'}{(x-\sqrt{1-x^2})^2(1-x^2)}$$

$$f''(x) = \frac{-\cancel{x^2} + 2x\sqrt{1-x^2} - 1 + \cancel{x^2} - (x+\sqrt{1-x^2})\cdot\left(1 + \frac{x}{\sqrt{1-x^2}}\right)\sqrt{1-x^2} - (x^2-1+x^2)\frac{-x}{\sqrt{1-x^2}}}{(x-\sqrt{1-x^2})^2(1-x^2)}$$

$$f''(x) = \frac{\cancel{2x\sqrt{1-x^2}} - 1 - \cancel{x^2} - \cancel{2x\sqrt{1-x^2}} - 1 + \cancel{x^2} + \frac{x(2x^2-1)}{\sqrt{1-x^2}}}{(x-\sqrt{1-x^2})^2(1-x^2)}$$

$$f''(x) = \frac{x(2x^2-1) - 2\sqrt{1-x^2}}{(x-\sqrt{1-x^2})^2(1-x^2)^{\frac{3}{2}}} = 0$$

$$x(2x^2-1) - 2\sqrt{1-x^2} = 0$$

$$2\sqrt{1-x^2} = x(2x^2-1) \quad /^2$$

$$x(2x^2-1) > 0 \quad \forall x \in \left(\frac{\sqrt{2}}{2}, 1\right)$$

$$4 - 4x^2 = 4x^6 - 4x^4 + x^2$$

$$4x^6 - 4x^4 + 5x^2 - 4 = 0 \quad (*)$$

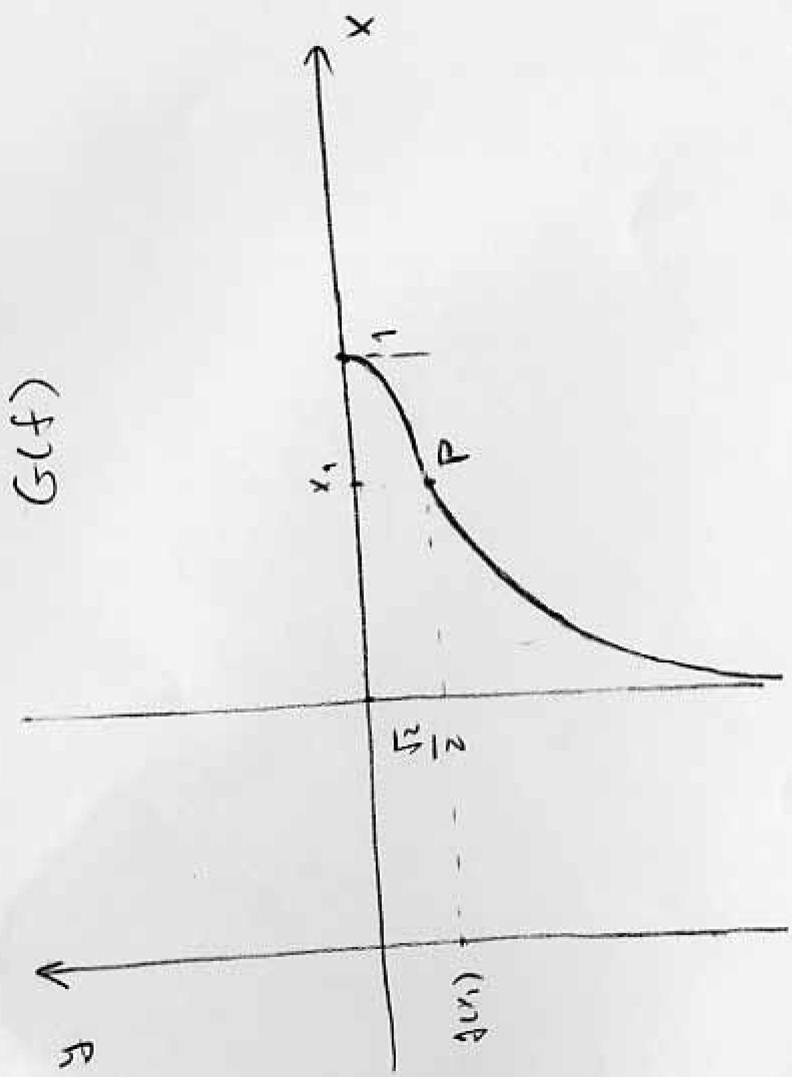
Jednačina (*) ima samo jedno realno rješenje $x_1 \approx 0{,}9063$ koje pripada intervalu $\left(\frac{\sqrt{2}}{2}, 1\right)$.

$P(x_1, f(x_1))$ – prevojna tačka

$f''(x) > 0$ za $x \in (x_1, 1) \Rightarrow f(x) \cup$

$f''(x) < 0$ za $x \in \left(\frac{\sqrt{2}}{2}, x_1\right) \Rightarrow f(x) \cap$

f) $Im(f)$: $y \in (-\infty, 0)$



Z A D A C I - Var. B :
za drugi parcijalni ispit iz IM1, 08. 01. 2007.

Zad. 1. Izračunajte (ili ustanovite da ne postoji) sljedeći limes funkcije koristeći asimptotsku relaciju $a^x = 1 + x\ln a + o(x) \quad (x \to 0), (a > 0)$:

$$\lim_{x\to 0}\left(\frac{3^x + 4^x + 5^x}{3}\right)^{\frac{1}{x}}.$$

I. 1.
II. $\sqrt[3]{60}$.
III. 12.
IV. Dati limes ne postoji.

Zad.2. Za realnu funkciju f jedne realne promjenljive zadanu formulom:

$$f(x) := \frac{1}{\sqrt{x\cdot(1+x)}}$$

ispitajte egzistenciju primitivne funkcije, a zatim izračunajte neodređeni integral $I(x) := \int f(x)dx$.

I. $I(x) = 2\operatorname{sgn} x \ln(\sqrt{|x|} + \sqrt{|x+1|}) + C, x \in \mathbf{R}\setminus[-1,0]$;

II. $I(x) = \operatorname{sgn} x \ln(\sqrt{|x|} + \sqrt{|x+1|}) + C, x \in \mathbf{R}\setminus[-1,0]$;

III. $I(x) = 2\operatorname{sgn} x (\sqrt{|x|} + \sqrt{|x+1|}) + C, x \in \mathbf{R}\setminus[-1,0]$;

IV. $I(x) = 2\operatorname{sgn} x \ln(\sqrt{|x|} \cdot \sqrt{|x+1|}) + C, x \in \mathbf{R}\setminus[-1,0]$.

Zad. 3. Izračunajte određeni integral

$$I := \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 e^{-x^2} \arctan\frac{1+x^2}{1-x^2}\, dx.$$

I. $I = -20$. II. $I = 20$.

III. I = 0. IV. $I = 10$.

Zad. 4. Izračunajte limes

$$l := \lim_{x \to +\infty} \frac{2^{2x^2}}{\int_9^x \left(2^{t^2} + 5\right) dt}.$$

I. $l = +\infty$.
II. $l = 1$.
III. $l = 0$.
IV. $l = 5$.

Zad. 5. Realna funkcija f jedne realne promjenljive zadana je formulom

$$f(x) := \left(\frac{x}{\varphi(x)}\right)^{\frac{2}{3}},$$

gdje je $\varphi(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (1 + x^2 \mathrm{tg}^2 t)^{-1} dt$

g) Odredite prirodni domen Dom (f), a zatim ispitajte ponašanje funkcije f na rubovima područja Dom (f) i odredite njene eventualne asimptote.

h) Odredite eventualne presjeke grafika G(f) sa koordinatnim osama i ispitajte znak zadane funkcije f.

i) Odredite eventualne tačke prekida i singulariteta i klasificirajte ih za zadanu funkciju f i njenu recipročnu funkciju $\frac{1}{f}$.

j) Odredite intervale monotonosti i eventualne tačke lokalnog i apsolutnog ekstrema zadane funkcije f, kao i eventualne prelomne i povratne tačke njenog grafika .

k) Ispitajte konveksnost i konkavnost i odredite eventualne prevojne tačke zadane funkcije f.

l) Odredite sliku Im(f) i nacrtajte grafik zadane funkcije f.

Rješenje:

Rješenja zadataka iz IM1 (Var B)

(Z1) $a^x = 1 + x\ln a + o(x) \quad (x \to 0), (a > 0)$

$$\lim_{x\to 0}\left(\frac{3^x+4^x+5^x}{3}\right)^{\frac{1}{x}}$$

Rješenje:

$$L = \lim_{x\to 0}\left(\frac{3^x+4^x+5^x}{3}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(\frac{1+x\ln 3+1+x\ln 4+1+x\ln 5}{3}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(1+x\ln\sqrt[3]{60}\right)^{\frac{1}{x}} = \lim_{x\to 0}\Bigg[\underbrace{\left(1+x\ln\sqrt[3]{60}\right)^{\frac{1}{x\ln\sqrt[3]{60}}}}_{\to e}\Bigg]^{\ln\sqrt[3]{60}} =$$

$$= e^{\ln\sqrt[3]{60}} = \sqrt[3]{60}$$

$$\boxed{L = \sqrt[3]{60}}$$

(Z2) $f(x) = \dfrac{1}{\sqrt{x(1+x)}}$, $\int f(x)\,dx$

Rješenje:

$$f(x) = \frac{1}{\sqrt{x(1+x)}}$$

$x(1+x) > 0$

	$-\infty$ … -1	-1 … 0	0 … $+\infty$
x	$-$	$-$	$+$
$1+x$	$-$	$+$	$+$
proizv.	$+$	$-$	$+$

$x \in (-\infty, -1) \cup (0, +\infty)$

F-ija $f(x)$ je definiciono neprekidna na $(-\infty, -1) \cup (0, +\infty)$

$$\int f(x)\,dx = \int \frac{dx}{\sqrt{x(1+x)}} = \left| \begin{array}{l} \sqrt{x^2+x} = x+t \quad \text{Eulerova smjena} \\ \cancel{x^2}+x = \cancel{x^2}+2xt+t^2 \\ x = \dfrac{t^2}{1-2t} \Rightarrow dx = \dfrac{2t(1-t)}{(1-2t)^2}dt \end{array} \right.$$

$$= \int \frac{\dfrac{2t\cancel{(1-t)}}{(1-2t)^{\cancel{2}}}}{\dfrac{t\cancel{(1-t)}}{\cancel{1-2t}}}dt = -2\int \frac{dt}{2t-1} = -\cancel{2}\cdot\frac{1}{\cancel{2}}\ln|2t-1| + C =$$

$$= -\ln|2t-1| + C = -\ln\left|2\sqrt{x^2+x} - 2x - 1\right| + C$$

(23) $$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} x^3 e^{-x^2} \operatorname{arctg}\frac{1+x^2}{1-x^2}\,dx$$

Rješenje:

$$f(x) = x^3 e^{-x^2} \operatorname{arctg}\frac{1+x^2}{1-x^2}$$

DP: $1-x^2 \neq 0$

$x \neq \pm 1$

$f(x)$ je neprekidna na segmentu $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

$f(-x) = -f(x) \rightarrow$ f-ija neparna

$$\Rightarrow I = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx = \int_{-\frac{1}{2}}^{0} f(x)\,dx + \int_{0}^{\frac{1}{2}} f(x)\,dx = \left| \begin{array}{l} \text{za 1. integral} \\ x=-t \\ dx=-dt \\ x=-\frac{1}{2} \to t=\frac{1}{2} \\ x=0 \to t=0 \end{array} \right.$$

$$= \int_{\frac{1}{2}}^{0} f(-t)(-dt) + \int_{0}^{\frac{1}{2}} f(x)\,dx = -\int_{\frac{1}{2}}^{0} (-f(t))\,dt + \int_{0}^{\frac{1}{2}} f(x)\,dx \overset{t=x}{=}$$

$$= -\int_{0}^{\frac{1}{2}} f(x)\,dx + \int_{0}^{\frac{1}{2}} f(x)\,dx = \underline{\underline{0}}$$

by Sejla Cebiric

24) $$L=\lim_{x\to\infty}\frac{2^{2x^2}}{\int_9^x (2^{t^2}+5)\,dt}$$

Rješenje:

Kako je $f(t)=2^{t^2}+5$ pozitivna i rastuća funkcija (definisana $\forall t\in\mathbb{R}$, pa kao kompozicija elementarnih funkcija i neprekidna $\forall t\in\mathbb{R}$) to će integral $\int_9^x (2^{t^2}+5)\,dt \to +\infty$ kada $x\to+\infty$.

Slijedi da će izraz u limesu biti neodređeni oblik tipa $\frac{\infty}{\infty}$ pa možemo primijeniti L'Hospitalov pravilo.

$$L=\lim_{x\to\infty}\frac{2^{2x^2}}{\int_9^x (2^{t^2}+5)\,dt}\overset{\text{PNO}}{\underset{\text{Lop.}}{\overset{\frac{\infty}{\infty}}{=}}}\lim_{x\to\infty}\frac{4\ln 2\cdot x\cdot 2^{2x^2}}{2^{x^2}+5}=$$

$$=4\ln 2\lim_{x\to\infty}\frac{\overbrace{x\cdot 2^{x^2}}^{\to+\infty}}{1+\underbrace{\frac{5}{2^{x^2}}}_{\to 0}}=+\infty$$

$$\boxed{L=+\infty}$$

by Sejla Cebiric

(25) $f(x) = \left(\frac{x}{\varphi(x)}\right)^{\frac{2}{3}}$

$$\varphi(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (1 + x^2 \operatorname{tg}^2 t)^{-1} dt \quad (*)$$

Rješenje:

U zadatku (23) varijanta A smo izračunali:

$$\varphi(x) = \frac{1}{|x|+1}, \text{ za } \forall x \in \mathbb{R} \setminus \{-1, 1\}$$

Nađimo $\varphi(x)$ u tačkama $x = 1$ i $x = -1$.

$$\varphi(1) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (1 + \operatorname{tg}^2 t)^{-1} dt = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^2 t}{\sin^2 t + \cos^2 t} dt =$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos^2 t\, dt = \frac{\cancel{2}}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos 2t + 1}{\cancel{2}} dt =$$

$$= \frac{1}{\pi}\left(\frac{1}{2}\sin 2t + t\right)\Big|_0^{\frac{\pi}{2}} = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}$$

Kako je f-ija $\varphi(x)$ data sa $(*)$ parna po x to će biti i $\varphi(-1) = \frac{1}{2}$.

Kako vrijedi

$$\lim_{x \to -1_-} \varphi(x) = \lim_{x \to -1_-} \frac{1}{|x|+1} = \lim_{x \to 1_+} \varphi(x) = \frac{1}{2} = \varphi(-1)$$

$$\lim_{x \to 1_-} \varphi(x) = \lim_{x \to 1_-} \frac{1}{|x|+1} = \lim_{x \to 1_+} \varphi(x) = \frac{1}{2} = \varphi(1)$$

to je $\varphi(x)$ neprekidna i u tačkama $x = 1$ i $x = -1$,

pa možemo uzeti da vrijedi $\varphi(x) = \frac{1}{|x|+1}$, $\forall x \in \mathbb{R}$.

Sada je $f(x)=\left[x(1+|x|)\right]^{\frac{2}{3}}=\sqrt[3]{x^2(1+|x|)^2}$

a) Dom(f): $x\in\mathbb{R}$, $f(x)$ neprekidna $\forall x\in\mathbb{R}$

Kako je $f(x)$ parna f-ija to bez umanjenja opštosti možemo dalje posmatrati f-iju $f(x)$ na intervalu $x\in[0,+\infty)$, te možemo pisati:

$$f(x):=\sqrt[3]{x^2(1+x)^2}$$

$$\lim_{x\to 0_+} f(x)=f(0)=0$$

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\sqrt[3]{x^2(1+x)^2}=+\infty$$

Moguća je K.AS. $y=kx+n$ za $x\to+\infty$

$$k=\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\sqrt{\underbrace{\left(\frac{1}{x}+1\right)}_{\to 1}\underbrace{(1+x)}_{\to+\infty}}=+\infty \Rightarrow$$

$\Rightarrow$ nema K. AS.

Zaključujemo da ni za $x\to-\infty$ nema K.AS.

b) $x=0\Rightarrow f(x)=0$ $\quad A(0,0)$

$f(x)=0\Rightarrow\sqrt[3]{x^2(1+x)^2}=0\Rightarrow x=0 \lor \cancel{x=-1}\notin[0,+\infty)$

ZNAK:

$f(x)>0\quad\forall x\in(0,+\infty)$

c) $f(x)$ je neprekidna $\forall x\in\mathbb{R}$

$$g(x)=\frac{1}{f(x)}=\frac{1}{\sqrt[3]{x^2(1+|x|)^2}},\quad x\in\mathbb{R}\setminus\{0\}$$

Očigledno je $g(x)$ neprekidna na svakom segmentu $[a,b] \in (-\infty,0)\cup(0,+\infty)$, a u tački $x=0$ f-ija ima singularitet.

$$\lim_{x\to 0-} g(x) = \lim_{x\to 0-} \frac{1}{\sqrt[3]{x^2(1+|x|)^2}} = +\infty$$

$$\lim_{x\to 0+} g(x) = \lim_{x\to 0+} \frac{1}{\sqrt[3]{x^2(1+|x|)}^2} = +\infty$$

Vidimo da je u $x=0$ singularitet tipa pola.

d) $f(x) = [x(1+x)]^{\frac{2}{3}}$

$$f'(x) = \frac{2}{3}[x(1+x)]^{-\frac{1}{3}}\cdot(1+2x) = \frac{2}{3}\cdot\frac{1+2x}{\sqrt[3]{x(1+x)}}, \quad x\in(0,+\infty)$$

$f'(x) > 0 \quad \forall x \in (0,+\infty) \Rightarrow f(x) \nearrow$

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0+} \frac{2}{3}\frac{1+2x}{\sqrt[3]{x(1+x)}} = +\infty \Rightarrow tg\,\alpha_{12L} = +\infty \Rightarrow$$

$$\Rightarrow \alpha_{12L} = \frac{\pi}{2}$$

Tačka $x=0$ je povratna tačka f-ije $f(x)$.

e) $$f''(x) = \frac{2}{3}\cdot\frac{2\sqrt[3]{x(1+x)} - (1+2x)\cdot\frac{1+2x}{3\sqrt[3]{x^2(1+x)^2}}}{\sqrt[3]{x^2(1+x)^2}}$$

$$f''(x) = \frac{2}{3}\,\frac{6x+6x^2-1-4x-4x^2}{3\sqrt[3]{x^4(1+x)^4}}$$

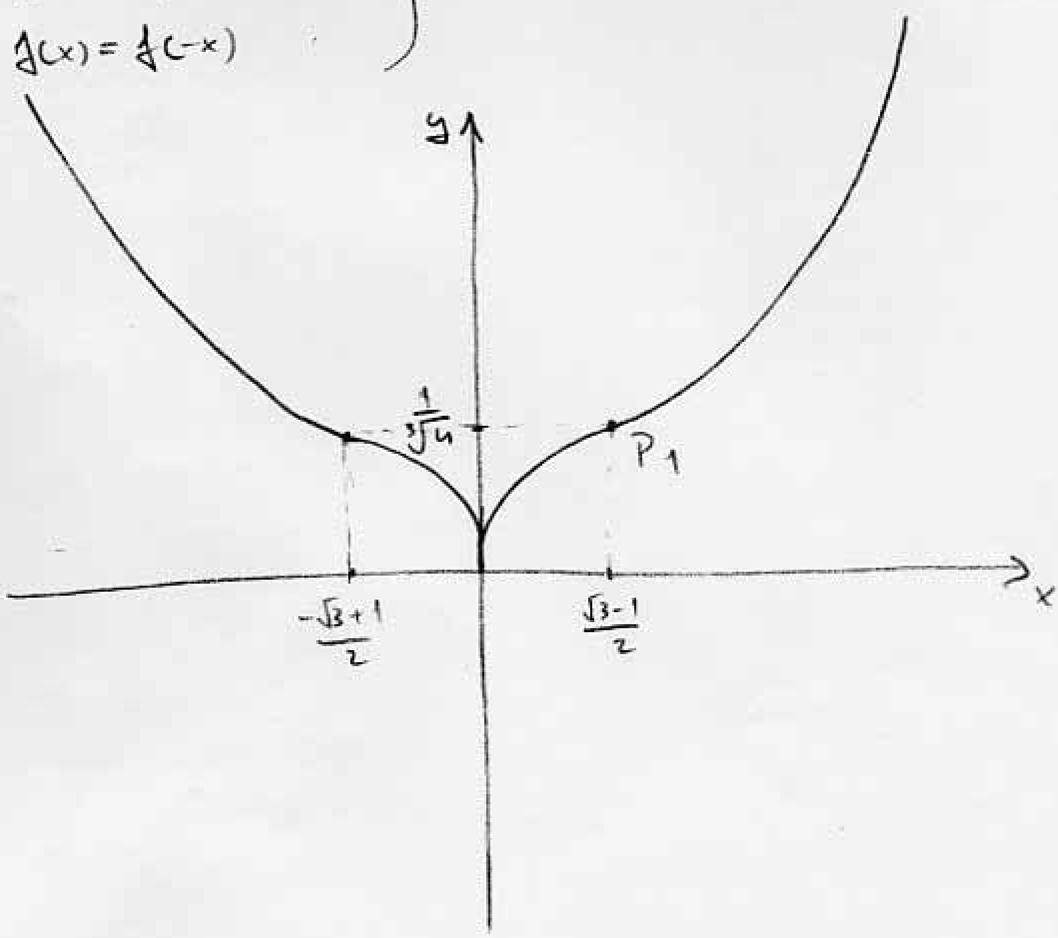
$$f''(x) = \frac{2}{9}\,\frac{2x^2+2x-1}{\sqrt[3]{x^4(1+x)^4}}, \quad x\in(0,+\infty)$$

$$2x^2+2x-1=0 \Rightarrow x_{1,2} = \frac{-2\pm\sqrt{4+8}}{4} = \frac{-2\pm2\sqrt{3}}{4}$$

$$x_1 = \frac{\sqrt{3}-1}{2} \qquad x_2 = \frac{-\sqrt{3}-1}{2} \notin (0,+\infty)$$

$$\Rightarrow \begin{array}{l} f''(x) > 0 \text{ za } x \in (x_1, +\infty) \Rightarrow f(x) \cup \\ f''(x) < 0 \text{ za } x \in (0, x_1) \Rightarrow f(x) \cap \quad P_1\left(\frac{\sqrt{3}-1}{2}, \frac{1}{\sqrt[3]{4}}\right) \end{array}$$

f) $$\left.\begin{array}{l} f(0) = 0 \\ \lim\limits_{x \to +\infty} f(x) = +\infty \\ f(x) \text{ neprekidna} \\ f(x) = f(-x) \end{array}\right\} \Rightarrow \operatorname{Im}(f):\ y \in [0, +\infty)$$



ZADACI - Var. C :
za drugi parcijalni ispit iz IM1, 08. 01. 2007.

Zad. 1. Izračunajte (ili ustanovite da ne postoji) sljedeći limes funkcije

$$\lim_{x\to 0}(\cos 2x)^{\frac{1}{2x^2}} .$$ Lop.

I. e.
II. 0.
III. Dati limes ne postoji.
IV. $\frac{1}{e}$.

Zad.2. Izračunajte derivaciju funkcije

$$f(x) := \int_{-x}^{x}\left(2^{t^2}+5\right)dt$$

u tački $x = 1$.

I. $f'(1) = 14$. II. $f'(1) = 7$.
III. $f'(1) = 2$. IV. $f'(1) = 8$.

Zad. 3. Kriva C zadana je jednačinom $xy^2 = 1 - x$. Izračunati površinu P lika ograničenog lukom krive C i tetivom koja spaja prevojne tačke (te krive).

I. $P = \frac{\pi}{2} - \frac{\sqrt{3}}{2}$. II. $P = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$.
III. $P = \frac{\pi}{2} - \frac{\sqrt{3}}{3}$. IV. $P = \frac{\pi}{3} - \frac{\sqrt{3}}{3}$.

Zad. 4. Nađite (prirodni) domen funkcije f zadane formulom

$$f(x) := \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1},$$

tj. odredite skup svih realnih brojeva x za koje zadani red konvergira.

I. $[0, +\infty)$.
II. $(1, +\infty)$.
III. $(-1, +\infty)$.
IV. $(0, +\infty)$.

Zad.5. Realna funkcija f jedne realne promjenljive zadana jeformulom

$$f(x) := \sqrt[3]{x^3 + 2x^2}\ .$$

m) Odredite prirodni domen Dom (f), a zatim ispitajte ponašanje funkcije f na rubovima područja Dom (f) i odredite njene eventualne asimptote.

n) Odredite eventualne presjeke grafika G(f) sa koordinatnim osama i ispitajte znak zadane funkcije f.

o) Odredite eventualne tačke prekida i singulariteta i klasificirajte ih za zadanu funkciju f i njenu recipročnu funkciju $\frac{1}{f}$.

p) Odredite intervale monotonosti i eventualne tačke lokalnog i apsolutnog ekstrema zadane funkcije f, kao i eventualne prelomne i povratne tačke njenog grafika .

q) Ispitajte konveksnost i konkavnost i odredite eventualne prevojne tačke zadane funkcije f.

r) Odredite sliku Im(f) i nacrtajte grafik zadane funkcije f.

by Sejla Cebiric Rješenja zadataka iz IM1 (Var C)

(Z1) $\lim\limits_{x\to 0} (\cos 2x)^{\frac{1}{2x^2}}$

Rješenje:

$$L = \lim_{x\to 0} (\cos 2x)^{\frac{1}{2x^2}} \quad /\circ \ln$$

$$\ln L = \lim_{x\to 0} \ln(\cos 2x)^{\frac{1}{2x^2}} = \lim_{x\to 0} \frac{\ln(\cos 2x)}{2x^2} \overset{\text{PNO } \frac{0}{0}}{\underset{\text{Lop.}}{=\!=}}$$

$$= \lim_{x\to 0} \frac{\frac{-\cancel{2}\sin 2x}{\cos 2x}}{\cancel{4}_2 x} = \lim_{x\to 0} \frac{-\sin 2x}{2x\cos 2x} =$$

$$= -\lim_{x\to 0} \underbrace{\frac{\sin 2x}{2x}}_{\to 1} \cdot \underbrace{\frac{1}{\cos 2x}}_{\to 1} = -1$$

$$\ln L = -1 \Rightarrow \boxed{L = e^{-1}}$$

(Z2) $f(x) = \int\limits_{-x}^{x} (2^{t^2} + 5)\,dt$, $\quad f'(1) = ?$

Rješenje:

$$f'(x) = (2^{x^2} + 5) + (2^{(-x)^2} + 5) = 2\cdot(2^{x^2} + 5)$$

$$f'(1) = 2\cdot(2+5)$$

$$\boxed{f'(1) = 14}$$

23) $xy^2 = 1 - x$

Rješenje:

$$y^2 = \frac{1-x}{x} \Rightarrow y = \pm\sqrt{\frac{1-x}{x}}$$

$$x \neq 0 \wedge \frac{1-x}{x} \geq 0 \Leftrightarrow x \in (0,1]$$

$$DP: \ x \in (0,1]$$

Naćimo prevojnu tačku f-ije $y = \sqrt{\frac{1-x}{x}}$ (možemo primijetiti da će to biti i prevojna tačka f-ije $y = -\sqrt{\frac{1-x}{x}}$).

$$y' = \frac{1}{2} \cdot \sqrt{\frac{x}{1-x}} \cdot \frac{-1}{x^2} \quad , \quad x \in (0,1)$$

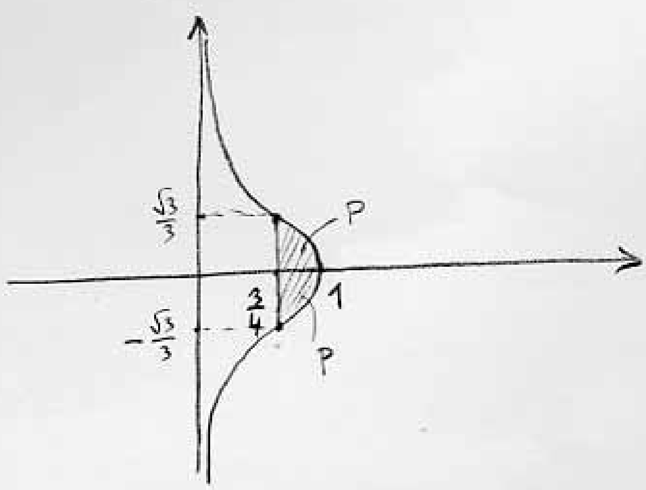
$$y'' = \frac{1}{2} \cdot \frac{1}{2}\sqrt{\frac{1-x}{x}} \cdot \frac{1}{(1-x)^2} \cdot \frac{-1}{x^2} + \frac{1}{\cancel{2}}\sqrt{\frac{x}{1-x}} \cdot \frac{\cancel{2}}{x^3} = 0$$

$$-\frac{1}{4x^2(1-x)^2}\sqrt{\frac{1-x}{x}} + \frac{1}{x^3}\sqrt{\frac{x}{1-x}} = 0 \ \Big/ \cdot 4x^3(1-x)^2\sqrt{x(1-x)}$$

$$-x(1-x) + 4(1-x)^2 x = 0$$

$$x(1-x)(-1+4-4x) = 0 \Rightarrow 3 - 4x = 0 \Rightarrow x_p = \frac{3}{4}$$

Skicirajmo grafik f-ije y:



Kako je grafik f-ije y simetričan u odnosu na osu Ox to je dovoljno naći površinu P između

...se Ox, $y=\sqrt{\frac{1-x}{x}}$ od tačke $x=\frac{3}{4}$ do $x=1$, te pomnožiti sa 2.

$$P=\int_{\frac{3}{4}}^{1} y(x)\,dx=\int_{0}^{\frac{\sqrt{3}}{3}}\left(x(y)-\frac{3}{4}\right)dy=\int_{0}^{\frac{\sqrt{3}}{3}}\left(\frac{1}{1+y^2}-\frac{3}{4}\right)dy$$

$$=\left(\operatorname{arctg} y-\frac{3}{4}y\right)\Big|_{0}^{\frac{\sqrt{3}}{3}}=\frac{\pi}{6}-\frac{\sqrt{3}}{4}$$

$$\boxed{2P=\frac{\pi}{3}-\frac{\sqrt{3}}{2}}$$

Z4 $$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}\left(\frac{x-1}{x+1}\right)^{2n-1}$$

Rješenje:

Isto rješenje kao u zadatku Z4 varijanta D.

5) by Sejla Cebiric $f(x)=\sqrt[3]{x^3+2x^2}$

Rješenje:

a) $Dom(f): \ x\in\mathbb{R}$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty}\sqrt[3]{x^2(x+2)}=-\infty$$

Moguća je kosa asimptota $y=kx+n$, $x\to-\infty$.

$$k=\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{\sqrt[3]{x^3+2x^2}}{x}=\lim_{x\to-\infty}\sqrt[3]{\frac{x^3+2x^2}{x^3}}=$$

$$=\lim_{x\to-\infty}\sqrt[3]{1+\frac{2}{x}}=1$$

$$n=\lim_{x\to-\infty}(f(x)-kx)=\lim_{x\to-\infty}\left(\sqrt[3]{x^3+2x^2}-x\right)=$$

$$=\lim_{x\to-\infty}\left(\sqrt[3]{x^3+2x^2}-\sqrt[3]{x^3}\right)\cdot\frac{\sqrt[3]{(x^3+2x^2)^2}+\sqrt[3]{(x^3+2x^2)x^3}+\sqrt[3]{(x^3)^2}}{\sqrt[3]{(x^3+2x^2)^2}+\sqrt[3]{(x^3+2x^2)x^3}+\sqrt[3]{(x^3)^2}}=$$

$$=\lim_{x\to-\infty}\frac{\cancel{x^3}+2x^2-\cancel{x^3}}{\sqrt[3]{(x^3+2x^2)^2}+\sqrt[3]{x^6+2x^5}+\sqrt[3]{x^6}}=$$

$$=\lim_{x\to-\infty}\frac{2x^2}{x^2\cdot\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+x^2\sqrt[3]{1+\frac{2}{x}}+x^2}=$$

$$=\lim_{x\to-\infty}\frac{2}{\sqrt[3]{\left(1+\frac{2}{x}\right)^2}+\sqrt[3]{1+\frac{2}{x}}+1}=\frac{2}{3}$$

K.AS. $y=x+\frac{2}{3}$, $x\to-\infty$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sqrt[3]{x^2(x+2)} = +\infty$$

Moguća je kosa asimptota $y = lx + m$, $x \to +\infty$

$$l = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \sqrt[3]{1+\frac{2}{x}} = 1$$

$$m = \lim_{x\to+\infty} (f(x) - lx) = \lim_{x\to+\infty} \left(\sqrt[3]{x^3+2x} - x\right) = \frac{2}{3}$$

K. AS. $y = x + \frac{2}{3}$, $x \to +\infty$

b) NULE F-ije

$$y = 0 \iff \sqrt[3]{x^2(x+2)} = 0 \iff x^2(x+2) = 0$$

$$x = 0 \quad \vee \quad x = -2$$

F-ija $f(x) = \sqrt[3]{x^2(x+2)}$ siječe Ox osu u tačkama $A(-2,0)$ i $B(0,0)$

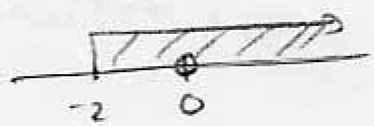
ZNAK F-ije

$$y > 0 \iff \sqrt[3]{x^2(x+2)} > 0 \iff x^2(x+2) > 0$$

$$x^2 > 0 \quad \forall x \in \mathbb{R} \setminus \{0\}$$

$$\Rightarrow x + 2 > 0$$

$$x > -2$$



$x \in (-2,0) \cup (0,+\infty)$

$$y < 0 \iff \sqrt[3]{x^2(x+2)} < 0 \iff x \in (-\infty,-2)$$

$y > 0$ za $x \in (-2,0) \cup (0,+\infty)$

$y < 0$ za $x \in (-\infty,-2)$

c) F-ja $f(x)$ je neprekidna $\forall x \in \mathbb{R}$.

$$g(x) = \frac{1}{f(x)} = \frac{1}{\sqrt[3]{x^2(x+2)}}$$

$Dom(g): \ x \in \mathbb{R} \setminus \{-2, 0\}$

$$\lim_{x \to -2_-} g(x) = \lim_{x \to -2_-} \frac{1}{\underbrace{\sqrt[3]{x^2(x+2)}}_{\to 0_-}} = -\infty$$

$$\lim_{x \to -2_+} g(x) = \lim_{x \to -2_+} \frac{1}{\underbrace{\sqrt[3]{x^2(x+2)}}_{\to 0_+}} = +\infty$$

$$\lim_{x \to 0_-} g(x) = \lim_{x \to 0_-} \frac{1}{\underbrace{\sqrt[3]{x^2(x+2)}}_{\to 0_+}} = +\infty$$

$$\lim_{x \to 0_+} g(x) = \lim_{x \to 0_+} \frac{1}{\underbrace{\sqrt[3]{x^2(x+2)}}_{\to 0_+}} = +\infty$$

Na osnovu prethodnog zaključujemo da je f-ja $g(x)$ neprekidna $\forall x \in \mathbb{R} \setminus \{-2, 0\}$, a u tačkama $x = -2$ i $x = 0$ ima esencijalni singularitet i singularitet tipa pola, respektivno.

d) $$f'(x) = \frac{3x^2 + 4x}{3\sqrt[3]{x^4(x+2)^2}} = \frac{3x+4}{3\sqrt[3]{x(x+2)^2}}, \quad x \in \mathbb{R} \setminus \{-2, 0\}$$

$$\lim_{x \to -2_-} f'(x) = \lim_{x \to -2_-} \frac{3x+4}{3\sqrt[3]{x(x+2)^2}} = +\infty$$

$$\lim_{x \to -2_+} f'(x) = \lim_{x \to -2_+} \frac{3x+4}{3\sqrt[3]{x(x+2)^2}} = +\infty$$

$\Rightarrow$ tačka $x=-2$ je povratna tačka f-ije $f(x)$.

$$\lim_{x\to 0_-} f'(x) = \lim_{x\to 0_-} \frac{3x+4}{3\sqrt[3]{x(x+2)^2}} = -\infty$$

$$\lim_{x\to 0_+} f'(x) = \lim_{x\to 0_+} \frac{3x+4}{3\sqrt[3]{x(x+2)^2}} = +\infty$$

$\Rightarrow$ tačka $x=0$ je povratna tačka f-ije $f(x)$

$$f'(x)=0 \Rightarrow \frac{3x+4}{3\sqrt[3]{x(x+2)^2}} = 0 \Rightarrow x=-\frac{4}{3}$$

	$-\infty$ … -2	-2 … $-\frac{4}{3}$	$-\frac{4}{3}$ … 0	0 … $+\infty$
$3x+4$	$-$	$-$	$+$	$+$
x	$-$	$-$	$-$	$+$
$(x+2)^2$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	$+$	$-$	$+$
$f(x)$	$\nearrow$	$\nearrow$	$\searrow$	$\nearrow$

$$f\left(-\frac{4}{3}\right) = \frac{2\sqrt[3]{4}}{3}$$

$$E\left(-\frac{4}{3}, \frac{2\sqrt[3]{4}}{3}\right) \quad \text{MAX}$$

e) $$f''(x) = \frac{1}{3}\cdot\frac{3\sqrt[3]{x(x+2)^2} - (3x+4)\cdot\frac{3x^2+8x+4}{3\sqrt[3]{x^2(x+2)^4}}}{\sqrt[3]{x^2(x+2)^4}}$$

$$f''(x) = \frac{9x(x+2)^2 - (3x+4)(3x^2+8x+4)}{9x(x+2)^2\sqrt[3]{x(x+2)^2}} = 0$$

$$f''(x) = \frac{\cancel{9x^3} + \cancel{36x^2} + 36x - \cancel{9x^3} - \cancel{24x^2} - 12x - \cancel{12x^2} - 32x - 16}{9\left[x(x+2)^2\right]^{\frac{4}{3}}}$$

by Sejla Cebiric

$$\frac{x-\sqrt{1-x^2}}{(x+\sqrt{1-x^2})\sqrt{1-x^2}} \cdot \frac{x+\sqrt{1-x^2}}{x+\sqrt{1-x^2}} = \frac{x^2-1+x^2}{(x^2+2x\sqrt{1-x^2}+1-x^2)\sqrt{1-x^2}} =$$

$x(x^2+4x+4)$

x^3+4x^2+4x

$$= \frac{1-2x\sqrt{1-x^2}}{(2x^2-1)\sqrt{1-x^2}}$$

$$\left(\frac{3x+4}{3\sqrt[3]{x(x+2)^2}}\right)' = \frac{(3x+4)'\sqrt[3]{x(x+2)^2}-(3x+4)\cdot\left[\sqrt[3]{x(x+2)^2}\right]'}{3\sqrt[3]{x^2(x+2)^4}} =$$

$$= \frac{3\sqrt[3]{x(x+2)^2}-(3x+4)\frac{3x^2+8x+4}{3\sqrt[3]{x^2(x+2)^4}}}{3\sqrt[3]{x^2(x+2)^4}} =$$

$$= \frac{9x(x+2)^2-(3x+4)(3x^2+8x+4)}{9\sqrt[3]{x^4(x+2)^8}} =$$

$$= \frac{\cancel{9x^3}+3\cancel{6x^2}+36x-\cancel{9x^3}-\cancel{24x^2}-12x-\cancel{12x^2}-32x-16}{9\sqrt[3]{x^4(x+2)^8}} =$$

$$= \frac{-8x-16}{}$$

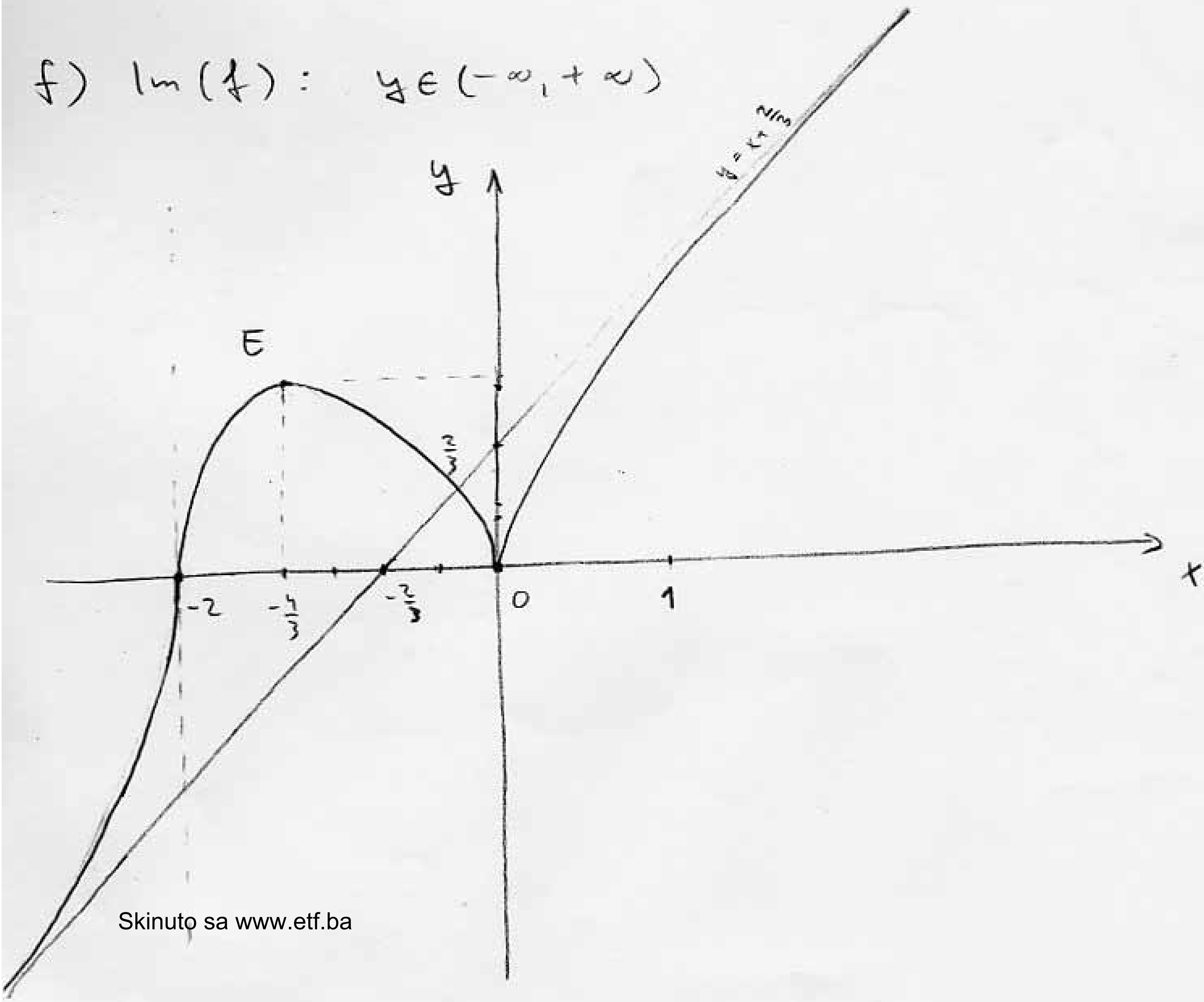
$$f''(x) = \frac{-8(x+2)}{9\left[x(x+2)^2\right]^{\frac{4}{3}}} = \frac{-8}{9\sqrt[3]{x^4(x+2)^5}}$$

$\Rightarrow$ nema prevojnih tačaka

$f''(x) > 0$ za $x \in (-2,0) \cup (0,+\infty) \Rightarrow f(x)$ $\cup$

$f''(x) < 0$ za $x \in (-\infty,-2) \Rightarrow f(x)$ $\cap$

f) $\text{Im}(f)$: $y \in (-\infty,+\infty)$



Z A D A C I - Var. D:
za drugi parcijalni ispit iz IM1, 08. 01. 2007.

Zad. 1. Za realnu funkciju f jedne realne promjenljive zadanu formulom $f(x) := \sqrt[3]{x^3 + 2x^2}$ odredite asimptotsku relaciju u slučaju $x \to \pm\infty$.

I. $$\sqrt[3]{x^3+2x^2} = x\left(1+\frac{2}{x}\right)^{\frac{1}{3}} = x + \frac{2}{3} - \frac{4}{9x} + o\left(\frac{1}{x}\right), \quad x \to \pm\infty .$$

II. $$\sqrt[3]{x^3+2x^2} = x\left(1+\frac{2}{x}\right)^{\frac{1}{3}} = x + 2 - \frac{4}{9x} + o\left(\frac{1}{x}\right), \quad x \to \pm\infty .$$

III. $$\sqrt[3]{x^3+2x^2} = x\left(1+\frac{2}{x}\right)^{\frac{1}{3}} = x + \frac{2}{3} - \frac{2}{3x} + o\left(\frac{1}{x}\right), \quad x \to \pm\infty .$$

IV. $$\sqrt[3]{x^3+2x^2} = x\left(1+\frac{2}{x}\right)^{\frac{1}{3}} = x + 2 - \frac{2}{3x} + o\left(\frac{1}{x}\right), \quad x \to \pm\infty .$$

Zad. 2. Za realnu funkciju f jedne realne promjenljive zadanu formulom:
$$f(x) := (a^2\cos^2 x + b^2 \sin^2 x)^{-1} \ (a,\ b \in \mathbf{R}\setminus\{0\})$$

nađite sve njene primitivne funkcije na njenom prirodnom domenu.

I. $$F(x) = \frac{1}{ab}\operatorname{arctg}(\operatorname{tg}x) + \frac{\pi n}{|ab|} + C \quad \text{za} \quad (2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2},$$
$$F\left((2n\pm1)\frac{\pi}{2}\right) = \frac{2n\pm1}{2|ab|} + C \quad \left(n = \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor\right).$$

II. $$F(x) = \frac{1}{a}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}x\right) + \frac{\pi n}{|ab|} + C \quad \text{za} \quad (2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2},$$
$$F\left((2n\pm1)\frac{\pi}{2}\right) = \frac{2n\pm1}{2|ab|} + C \quad \left(n = \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor\right).$$

III. $$F(x) = \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}x\right) + \frac{\pi n}{|ab|} + C \quad \text{za} \quad (2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2},$$
$$F\left((2n\pm1)\frac{\pi}{2}\right) = \frac{2n\pm1}{2|ab|} + C \quad \left(n = \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor\right).$$

IV. $$F(x) = \frac{b}{a}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}x\right) + \frac{\pi n}{|ab|} + C \quad \text{za} \quad (2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2},$$
$$F\left((2n\pm1)\frac{\pi}{2}\right) = \frac{2n\pm1}{2|ab|} + C \quad \left(n = \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor\right).$$

Zad. 3. Izračunajte integral

$$I := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2 x \cos x}\, dx .$$

I. $I = \frac{2}{3}$. II. $I = \frac{4}{3}$.

III. $I = \frac{2}{3}\pi$. IV. $I = \frac{4}{3}\pi$.

Zad. 4. Nađite skup svih realnih brojeva x za koje konvergira red

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{x-1}{x+1} \right)^{2n+1} .$$

I. $[0, +\infty)$.
II. $(1, +\infty)$.
III. $(-1, +\infty)$.
IV. $(0, +\infty)$.

Zad. 5. Realna funkcija f jedne realne promjenljive zadana je formulom

$$f(x) := \operatorname{sgn}(x) \cdot \ln(x + \sqrt{n^2 - x^2}),$$

gdje je n najmanja cifra Vašeg jedinstvenog matičnog broja koja je veća od 0.

s) Odredite prirodni domen $\operatorname{Dom}(f)$ i ispitajte ponašanje funkcije f na rubovima područja $\operatorname{Dom}(f)$.
t) Odredite eventualne presjeke grafika $G(f)$ sa koordinatnim osama i ispitajte znak zadane funkcije f.
u) Odredite eventualne tačke prekida i singulariteta i klasificirajte ih za zadanu funkciju f i njenu recipročnu funkciju $\frac{1}{f}$.
v) Odredite intervale monotonosti i eventualne tačke lokalnog i apsolutnog ekstrema zadane funkcije f, kao i eventualne prelomne i povratne tačke grafika njene recipročne funkcije $\frac{1}{f}$.
w) Ispitajte konveksnost i konkavnost i odredite eventualne prevojne tačke zadane funkcije f.
x) Odredite sliku $\operatorname{Im}(f)$ i nacrtajte grafik zadane funkcije f.

Rješenja zadataka iz IM1 (Var. D)

(Z1) $f(x) = \sqrt[3]{x^3+2x^2}$, $x \to +\infty$ i $x \to -\infty$

Rješenje:

DP: $\forall x \in \mathbb{R}$

$$f(x) = x\left(1+\frac{2}{x}\right)^{\frac{1}{3}}$$

smjena: $x = \frac{1}{t}$ za $x \to +\infty$ (ili $-\infty$) $t \to 0_+$ (0_-)

$$\left(1+\frac{2}{x}\right)^{\frac{1}{3}} \equiv (1+2t)^{\frac{1}{3}} = \varphi(t)$$

Razvijmo f-iju $\varphi(t)$ u Maclaurentov red:

$$\varphi'(t) = \frac{2}{3}(1+2t)^{-\frac{2}{3}} \Rightarrow \varphi'(0) = \frac{2}{3}$$

$$\varphi''(t) = -\frac{8}{9}(1+2t)^{-\frac{5}{3}} \Rightarrow \varphi''(0) = -\frac{8}{9}$$

$$\vdots$$

$$\varphi^{(k)}(t) = -\left(-\frac{2}{3}\right)^k \cdot 2 \cdot 5 \cdot \ldots \cdot (3k-1) \cdot (1+2t)^{-\frac{3k-1}{3}}$$

Sada je:

$$\varphi(t) = 1 + \frac{2}{3}t - \frac{4}{9}t^2 + o(t^3) \quad , \ t \to 0$$

$$\frac{1}{t}\cdot\varphi(t) = \frac{1}{t} + \frac{2}{3} - \frac{4}{9}t + o(t^2) \quad , \ t \to 0$$

vraćamo smjenu $t = \frac{1}{x}$

$$f(x) = x + \frac{2}{3} - \frac{4}{9x} + o\left(\frac{1}{x^2}\right), \quad x \to +\infty \text{ (ili } x \to -\infty)$$

22) $f(x) = \dfrac{1}{a^2\cos^2 x + b^2\sin^2 x}$, $(a, b \in \mathbb{R}\setminus\{0\})$

Rješenje:

Kako je $a^2\cos^2 x + b^2 \sin^2 x \neq 0 \quad \forall x \in \mathbb{R}$ i $a, b \in \mathbb{R}\setminus\{0\}$ to je $f(x)$ def.-nep. $\forall x \in \mathbb{R}$ i $a, b \in \mathbb{R}\setminus\{0\}$, tj. $f(x)$ je neprekidna na svakom segmentu $[a, b] \in (-\infty, \infty)$ pa je i integrabilna.

$$I = \int f(x)\,dx = \int \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}\,dx = \frac{1}{a^2}\int \frac{\frac{1}{\cos^2 x}\,dx}{1 + \left(\frac{b}{a}\,\mathrm{tg}\,x\right)^2} =$$

$$= \left| \begin{array}{l} \frac{b}{a}\,\mathrm{tg}\,x = t \\ \frac{b}{a}\frac{dx}{\cos^2 x} = dt \\ \frac{dx}{\cos^2 x} = \frac{a}{b}\,dt \end{array} \right| = \frac{1}{a^{\cancel{2}}} \cdot \frac{\cancel{a}}{b} \int \frac{dt}{1 + t^2} =$$

$$= \frac{1}{ab}\,\mathrm{arctg}\,t + C = \frac{1}{ab}\,\mathrm{arctg}\,\frac{b}{a}\,\mathrm{tg}\,x + C$$

by Sejla Cebiric

23) $$I=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sqrt{\sin^2 x\cos x}\,dx$$

Rješenje:

$f(x)=\sqrt{\sin^2 x\cos x}$, $\sin^2 x\cos x \geq 0 \quad \forall x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$f(x)$ definisana $\forall x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, pa je i neprekidna na $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$. Slijedi da je i integrabilna na $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$.

$$f(-x)=f(x) \Rightarrow I=2\int_0^{\frac{\pi}{2}}\sqrt{\sin^2 x\cos x}\,dx =$$

$$=2\int_0^{\frac{\pi}{2}}|\sin x|\sqrt{\cos x}\,dx = 2\int_0^{\frac{\pi}{2}}\sin x\sqrt{\cos x}\,dx =$$

$$=\left|\begin{array}{l}\cos x = t^2\\ -\sin x\,dx = 2t\,dt\\ x=\frac{\pi}{2}\rightarrow t=0\\ x=0\rightarrow t=1\end{array}\right| = -4\int_1^0 t^2\,dt = -4\cdot\frac{t^3}{3}\Big|_1^0 = \frac{4}{3}$$

24) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{x-1}{x+1}\right)^{2n+1} \quad (*)$$

Rješenje:

$x\neq -1$

$$\frac{x-1}{x+1}=t \Rightarrow \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\,t^{2n+1} \quad (**)$$

$$a_n=\frac{(-1)^n}{2n+1}$$

$$\frac{1}{R}=\lim_{n\to\infty}\sqrt[n]{|a_n|}=\lim_{n\to\infty}\sqrt[n]{2n+1}=1 \Rightarrow R=1$$

$\Rightarrow$ red $(**)$ je k za $\forall\ |t|<1$ a uniformno k za $\forall\ |t|\le \rho<1$.

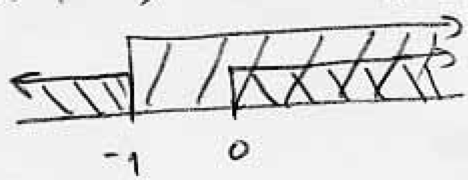
$\Rightarrow$ red $(*)$ je k za $\left|\frac{x-1}{x+1}\right|<1$

$$-1<\frac{x-1}{x+1}<1$$

$\frac{x-1}{x+1}>-1$ $\qquad\qquad$ $\frac{x-1}{x+1}<1$

$\frac{2x}{x+1}>0$ $\qquad\qquad$ $\frac{1}{x+1}>0$

$x\in(-\infty,-1)\cup(0,+\infty)$ $\qquad\qquad$ $x\in(-1,+\infty)$



$x\in(0,+\infty)$

Za $x\in(0,+\infty)$ red $(*)$ je k.

Ispitajmo šta je sa k reda $(*)$ za $\left|\frac{x-1}{x+1}\right|=1$

1° $\frac{x-1}{x+1}=-1 \iff x=0$

red $(*)$ postaje $\sum\limits_{n=0}^{\infty}\frac{(-1)^{n+1}}{2n+1}$. Kako $\frac{1}{2n+1}$ monotono teži ka nuli to je prema Leibnitzovom kriteriju red $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{2n+1}$ konvergentan

2° $\frac{x-1}{x+1}=1 \iff -1=1$ što je nemoguće

Dakle red $(*)$ je k za $\forall\ x\in[0,+\infty)$.

5) $f(x) = \operatorname{sgn}(x) \cdot \ln\left(x + \sqrt{m^2 - x^2}\right)$

Rješenje:

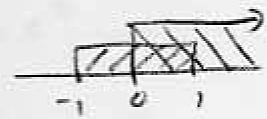
$m = 1 \Rightarrow f(x) = \operatorname{sgn}(x) \cdot \ln\left(x + \sqrt{1 - x^2}\right)$

a) $1 - x^2 \geq 0$ $\qquad$ $x + \sqrt{1-x^2} > 0$

$|x| \leq 1$ $\qquad$ $\sqrt{1-x^2} > -x$

1° $|x| \leq 1 \wedge -x \leq 0$

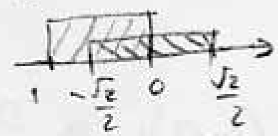
$x \geq 0$



R_1: $x \in [0, 1]$

2° $|x| \leq 1 \wedge -x > 0$ $\cap \Rightarrow x \in [-1, 0)$

$1 - x^2 > x^2 \Leftrightarrow x^2 < \frac{1}{2} \Leftrightarrow |x| < \frac{\sqrt{2}}{2}$



R_2: $x \in \left(-\frac{\sqrt{2}}{2}, 0\right)$

$R_1 \cup R_2$: $x \in \left(-\frac{\sqrt{2}}{2}, 1\right]$

$Dom(f)$: $x \in \left(-\frac{\sqrt{2}}{2}, 1\right]$

$$\lim_{x \to -\frac{\sqrt{2}}{2}+} f(x) = \lim_{x \to -\frac{\sqrt{2}}{2}+} \left[-\ln\left(x + \sqrt{1-x^2}\right)\right] = +\infty$$

V. AS. $\quad x = -\frac{\sqrt{2}}{2}+$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \ln\left(x + \sqrt{1-x^2}\right) = 0$$

$f(1) = 0$

F-ija $f(x)$ je u tački $x = 1$ neprekidna s lijeva.

) za $x=0 \Rightarrow y=0$ presjek sa Oy osom

NULE:

$$f(x)=0 = \ln(x+\sqrt{1-x^2}) \Rightarrow x+\sqrt{1-x^2}=1$$

$$\sqrt{1-x^2}=1-x \quad /^2$$

$$1-x \geq 0 \Leftrightarrow x \leq 1 \quad (T)$$

$$\cancel{1}-x^2=\cancel{1}-2x+x^2$$

$$2x^2-2x=0$$

$$2x(x-1)=0$$

$$x=0 \qquad x=1$$

ZNAK:

	$-\frac{\sqrt{2}}{2}$	0	1
$sgn(x)$	$-$	$+$	
$\ln(x+\sqrt{1-x^2})$	$-$	$+$	
$f(x)$	$+$	$+$	

$$\Rightarrow f(x)>0 \quad \forall x \in (-\tfrac{\sqrt{2}}{2},0)\cup(0,1)$$

c) F-ija $f(x)$ je neprekidna $\forall x \in (-\frac{\sqrt{2}}{2}, 1]$.

$$g(x)=\frac{1}{f(x)}=\frac{1}{sgn(x)\cdot \ln(x+\sqrt{1-x^2})}$$

$g(x)$ je definisana i neprekidna $\forall x \in (-\frac{\sqrt{2}}{2},0)\cup(0,1)$.

Tačka $x=0$ je singularitet f-ije $g(x)$

$$\left.\begin{array}{l} \lim\limits_{x\to 0-} g(x)=+\infty \\ \lim\limits_{x\to 0+} g(x)=+\infty \end{array}\right\} \Rightarrow x=0 \text{ je singularitet tipa pola}$$

1)
$$f(x)=\begin{cases}-\ln(x+\sqrt{1-x^2}), & x\in\left(-\frac{\sqrt2}{2},0\right)\\ 0, & x=0\\ \ln(x+\sqrt{1-x^2}), & x\in(0,1]\end{cases}$$

1° $x\in\left(-\frac{\sqrt2}{2},0\right)$

$$f'(x)=\frac{-1}{x+\sqrt{1-x^2}}\left(1-\frac{x}{\sqrt{1-x^2}}\right)=\frac{-\sqrt{1-x^2}+x}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}$$

$$f'(x)=0 \Rightarrow -\sqrt{1-x^2}+x=0 \Rightarrow x\in\emptyset \text{ jer je }$$
$$-\sqrt{1-x^2}+x<0 \quad \forall x\in\left(-\tfrac{\sqrt2}{2},0\right)$$

$$\Rightarrow f'(x)<0 \quad \forall x\in\left(-\tfrac{\sqrt2}{2},0\right) \Rightarrow f(x)\searrow \quad \forall x\in\left(-\tfrac{\sqrt2}{2},0\right)$$

$$\lim_{x\to 0-} f'(x)=\lim_{x\to 0-}\frac{x-\sqrt{1-x^2}}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}=-1 \Rightarrow \operatorname{tg}\alpha_{UL}=-1$$
$$\Rightarrow \alpha_{UL}=\frac{3\pi}{4}$$

$$f'(0-)=-1$$

3° $x\in(0,1)$

$$f'(x)=\frac{-x+\sqrt{1-x^2}}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}=0 \Rightarrow -x+\sqrt{1-x^2}=0$$
$$x_1=\frac{\sqrt2}{2} \qquad \cancel{x_2=-\frac{\sqrt2}{2}}$$

	0	$\frac{\sqrt2}{2}$	1
$f'(x)$	$+$	0	$-$
$f(x)$	$\nearrow$		$\searrow$

$$\Rightarrow x=\frac{\sqrt2}{2} \text{ MAX} \qquad f\left(\frac{\sqrt2}{2}\right)=\ln\sqrt2$$

by Sejla Cebiric

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\sqrt{1-x^2}-x}{(x+\sqrt{1-x^2})\sqrt{1-x^2}} = 1 \Rightarrow tg\,\alpha_{IZL} = 1 \Rightarrow$$
$$\Rightarrow \alpha_{IZL} = \frac{\pi}{4}$$

U tački $x=0$ imamo prelomnu tačku f-ije $f(x)$.

$$g(x) = \frac{1}{f(x)} = \begin{cases} \dfrac{-1}{\ln(x+\sqrt{1-x^2})}, & x \in \left(-\frac{\sqrt{2}}{2}, 0\right) \\ \dfrac{1}{\ln(x+\sqrt{1-x^2})}, & x \in (0,1) \end{cases}$$

1° $x \in \left(-\frac{\sqrt{2}}{2}, 0\right)$

$$g'(x) = \frac{-1}{\ln^2(x+\sqrt{1-x^2})} \cdot \frac{x-\sqrt{1-x^2}}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}$$

2° $x \in (0,1)$

$$g'(x) = \frac{-1}{\ln^2(x+\sqrt{1-x^2})} \cdot \frac{\sqrt{1-x^2}-x}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}$$

F-ija $g(x)$ nema prelomnih i povratnih tačaka.

e) $$f'(x) = \begin{cases} \dfrac{x-\sqrt{1-x^2}}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}, & x \in \left(-\frac{\sqrt{2}}{2}, 0\right) \\ \dfrac{\sqrt{1-x^2}-x}{(x+\sqrt{1-x^2})\sqrt{1-x^2}}, & x \in (0,1) \end{cases}$$

1° $x \in \left(-\frac{\sqrt{2}}{2}, 0\right)$

$$f''(x) = \frac{\left(1+\frac{x}{\sqrt{1-x^2}}\right)(x+\sqrt{1-x^2})\sqrt{1-x^2} - (x-\sqrt{1-x^2})\left[(x+\sqrt{1-x^2})\sqrt{1-x^2}\right]'}{(x+\sqrt{1-x^2})^2(1-x^2)}$$

$$f''(x) = \frac{(x+\sqrt{1-x^2})^2 - (x-\sqrt{1-x^2})(1-\frac{x}{\sqrt{1-x^2}})\sqrt{1-x^2} - (x^2-1+x^2)\frac{-x}{\sqrt{1-x^2}}}{(x+\sqrt{1-x^2})^2(1-x^2)}$$

$$f''(x) = \frac{\cancel{x^2}+\cancel{2x\sqrt{1-x^2}}+1-\cancel{x^2}+\cancel{x^2}-\cancel{2x\sqrt{1-x^2}}+1-\cancel{x^2}+\frac{x(2x^2-1)}{\sqrt{1-x^2}}}{(x+\sqrt{1-x^2})^2(1-x^2)}$$

$$f''(x) = \frac{x(2x^2-1)+2\sqrt{1-x^2}}{(x+\sqrt{1-x^2})^2(1-x^2)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \quad x(2x^2-1)+2\sqrt{1-x^2} = 0$$

$$2\sqrt{1-x^2} = x(2x^2-1) \quad /^2$$

$$x(2x^2-1) > 0 \iff$$

$$2x(x-\frac{\sqrt{2}}{2})(x+\frac{\sqrt{2}}{2}) > 0 \quad T \quad \forall x \in (-\frac{\sqrt{2}}{2}, 0)$$

$$4-4x^2 = 4x^6-4x^4+x^2$$

$$4x^6-4x^4+5x^2-4 = 0 \quad (*)$$

Jednačina $(*)$ ima samo jedno realno rješenje $x_1 \approx 0{,}936$. koji ne pripada $(-\frac{\sqrt{2}}{2}, 0)$. Za $x < x_1$ je $f''(x) > 0 \Rightarrow \cup$

$2^\circ \quad x \in (0, 1)$

$$f''(x) = -\frac{x(2x^2-1)+2\sqrt{1-x^2}}{(x+\sqrt{1-x^2})^2(1-x^2)^{\frac{3}{2}}} = 0 \Rightarrow$$

$$\Rightarrow \quad x(2x^2-1)+2\sqrt{1-x^2} = 0$$

$$2\sqrt{1-x^2} = x(2x^2-1) \quad /^2$$

$$x(2x^2-1) > 0 \quad \cap \; DP \Rightarrow x \in (\frac{\sqrt{2}}{2}, 1)$$

$$4x^6-4x^4+5x^2-4 = 0 \quad (**)$$

$x_2 \approx 0{,}936$ je jedino realno rješenje jednačine $(**)$ i $x_2 \in (\frac{\sqrt{2}}{2}, 1)$.

$f''(x) < 0$ za $\forall x \in (0,1) \Rightarrow f(x) \frown$

f) $\left.\begin{array}{l} f(1) = 0 \\ f(-\frac{\sqrt{2}}{2}+) = +\infty \\ f(x) \text{ - neprekidna na } Dom(f) \\ f(x) > 0 \quad \forall x \in Dom(f) \end{array}\right\} \Rightarrow Im(f): y \in [0, +\infty)$