Z A D A C I - Var. A :
za drugi parcijalni ispit iz IM1, 09. 01. 2008.

Zad. 1. Aproksimirajte funkciju f zadanu formulom $f(x) := 1 + \sqrt{x^5 + x^4}$ *Maclaurinovim polinomom* četvrtog stepena i procijenite grešku $|R_4|$ za $x \in \left[0, \frac{1}{2}\right]$.

I. $f(x) \approx 1 + x + \frac{1}{2}x^3 - \frac{1}{8}x^4$, $|R_4| < \frac{1}{2^4}$. **III.** $f(x) \approx 1 + x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4$, $|R_4| < \frac{1}{2^9}$.

II. $f(x) \approx 1 + x^2 + \frac{1}{2}x^3 - \frac{1}{3}x^4$, $|R_4| < \frac{1}{2^5}$. **IV.** $f(x) \approx 1 + x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4$, $|R_4| < \frac{1}{2^8}$.

Zad. 2. Izračunajte integrale $I := \int \left(\frac{1}{\sin x} + \frac{1}{\operatorname{sh} x} \right) dx$, $J := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2 x \cos x}\, dx$.

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I. $I = \ln\left|\sin x \operatorname{th}\frac{x}{2}\right| + C$, $J = \frac{2}{3}$. **III.** $I = \ln\left|\operatorname{tg}\frac{x}{2} \operatorname{th}\frac{x}{2}\right| + C$, $J = \frac{4}{3}$.

II. $I = \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right) \operatorname{th}\frac{x}{2}\right| + C$, $J = \frac{2}{3}\pi$. IV. $I = \frac{1}{2}\left[\ln\left|\operatorname{tg}\frac{x}{2}\right| + \ln\left|\operatorname{th}\frac{x}{2}\right|\right] + C$, $J = \frac{4}{3}\pi$.

Zad. 3. Izračunajte derivaciju funkcije $f(x) := \int_{-x}^{x} \frac{t^2 + 1}{t - 1} dt$ u tački $x = \frac{1}{2}$.

I. $f'\left(\frac{1}{2}\right) = \frac{10}{3}$. **II.** $f'\left(\frac{1}{2}\right) = -\frac{10}{3}$. III. $f'\left(\frac{1}{2}\right) = \frac{2}{3}$. IV. $f'\left(\frac{1}{2}\right) = -\frac{2}{3}$.

Zad. 4. Kriva C zadana je jednačinom $xy^2 = 8 - 4x$. Izračunati zapreminu tijela koje nastaje rotacijom oko x-ose lika (u xy-ravni) ograničenog zadanom krivom C i pravom koja prolazi kroz prevojne tačke te krive.

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I. $V = \frac{8\pi}{3} - \frac{\sqrt{3}}{2}$. **II.** $V = 8\pi\left(\ln\frac{4}{3} - \frac{1}{4}\right)$. III. $V = 8\pi\left(\ln\frac{2}{3} - 1\right)$. IV. $V = 8\pi\left(\ln\frac{2}{3} - \frac{1}{2}\right)$.

Zad. 5. Realna funkcija f jedne realne promjenljive zadana je formulom $f(x) := \operatorname{arc\,ctg}\frac{x+n}{x-n}$, gdje je n najmanja cifra Vašeg jedinstvenog matičnog broja koja je veća od 1.

a) Odredite prirodni domen Dom (f), a zatim ispitajte ponašanje funkcije f na rubovima područja Dom (f) i odredite njene eventualne asimptote.
b) Odredite eventualne presjeke grafika G(f) sa koordinatnim osama i ispitajte znak zadane funkcije f.
c) Odredite eventualne tačke prekida i singulariteta i klasificirajte ih za zadanu funkciju f i njenu recipročnu funkciju $\frac{1}{f}$.
d) Odredite intervale monotonosti i eventualne tačke lokalnog i apsolutnog ekstrema zadane funkcije f, kao i eventualne prelomne i povratne tačke njenog grafika .
e) Ispitajte konveksnost i konkavnost i odredite eventualne prevojne tačke zadane funkcije f.
f) Odredite sliku Im(f) i nacrtajte grafik zadane funkcije f.

Rješenje: ....@..................

Zadatak 1

$$f(x) = 1 + \sqrt{x^5 + x^4} = 1 + /$$

Po Maclaurenovom razvoju funkcije u polinomski oblik je

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!} + x^3 + \dots$$

Za zadanu funkciju je:

$$f(x) = 1 + \sqrt{x^4\left(1 + \frac{1}{x}\right)} = 1 + x^2\sqrt{1+x}$$

Neka je $\sqrt{\ }$ $g(t)$, i tu funkciju razvijmo po Mcl. razvoju

$$g(t) = \sqrt{1+t}$$

$$g(0) = 1$$

$$g'(t) = (1+t)^{\frac{1}{2}} = \frac{1}{2}(1+t)^{-\frac{1}{2}}; \quad g'(0) = \frac{1}{2}$$

$$g''(t) = \frac{1}{2}\cdot\left(-\frac{1}{2}\right)(1+t)^{-\frac{3}{2}}; \quad g''(0) = -\frac{1}{4}$$

$$g'''(t) = -\frac{1}{4}\cdot\left(\frac{-3}{2}\right)(1+t)^{-\frac{5}{2}}; \quad g^{(3)}(0) = \frac{3}{8}$$

$$g^{(IV)}(t) = \frac{3}{8}\cdot\left(-\frac{5}{2}\right)(1+t)^{-\frac{7}{5}}; \quad g^{(IV)}(0) = -\frac{15}{16}$$

pa je po ... razvoju

$$g(t) = 1 + \frac{1}{2}t - \frac{\frac{1}{4}}{2!}t^2 + \frac{\frac{3}{8}}{3!}t^3 - \frac{\frac{15}{16}}{4!}t^4 + R(t^5)$$

$$g(t) = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \frac{1}{16}t^3 - 15 \cdot \ldots + R(t^5)$$

pa je

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{15}{2}x^4 + R(x^5) \Big/ \cdot x^2$$

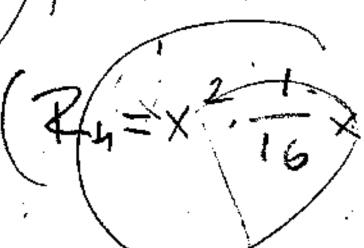
$$x^2\sqrt{1+x} = x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4 + x^2 R(x^3) \Big/ + 1$$

$$1 + x^2\sqrt{1+x} \approx 1 + x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4 \quad \left(R_4 = x^2 \cdot \frac{1}{16}x^3\right.$$

$$f(x) \approx 1 + x^2 + \frac{1}{2}x^3 - \frac{1}{8}x^4$$

Sada procjenimo R_4, za $x \in [0, \frac{1}{2}]$

$$R_4 = \frac{1}{16}x^5$$



Kako je $0 \le x \le \frac{1}{2}$, to je

$$0 \le x^5 \le \frac{1}{2^5} \Big/ \cdot \frac{1}{16} \iff$$

$$0 \le \frac{1}{16}x^5 \le \frac{1}{2^5} \cdot \frac{1}{2^4} \iff 0 \le R_4 \le \frac{1}{2^9}$$

pa će ... biti $R_4 < \frac{1}{2^9}$

Zadatak 2.

$$I = \int\left(\frac{1}{\sin x} + \frac{1}{\operatorname{sh} x}\right)dx \qquad \operatorname{sh} = \frac{e^x - e^{-x}}{2}$$

$$I = \underbrace{\int \frac{dx}{\sin x}}_{I_1} + \underbrace{\int\frac{dx}{\operatorname{sh} x}}_{I_2} \Longleftrightarrow I_1 = \int\frac{dx}{\sin x}\ ;\ I_2 = \int\frac{dx}{\operatorname{sh} x}\ i$$

$$\frac{e^x - \frac{1}{e^x}}{e} \qquad \frac{\frac{e^x - e^{-x}}{2}}{\frac{e^x + e^{-x}}{2}} \qquad \frac{e^x - \frac{1}{e^x}}{e^x + \frac{1}{e^x}}$$

$$I = I_1 + I_2$$

Izračunajmo integrale I_1 i I_2

$$I_1 = \int\frac{dx}{\sin x} = \left| \begin{array}{l} \sin x = \frac{\operatorname{tg}^2\frac{x}{2} - 1}{1 + \operatorname{tg}^2\frac{x}{2}}\ ;\ \sin x = \frac{t^2 - 1}{1+t^2} \\ \operatorname{tg}\frac{x}{2} = t\ ;\ \frac{dx}{\cos^2\frac{x}{2}} = dt \end{array}\right. \qquad dx = \cos^2\frac{x}{2}\,dt \qquad \cos^2\frac{x}{2} = \frac{1}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1}{1+t^2}$$

$$\left. dx = \frac{dt}{1+t^2} \right| = \int \frac{\frac{dt}{\cancel{1+t^2}}}{\frac{t^2-1}{\cancel{1+t^2}}} = \int\frac{dt}{\cancel{t^2-1}} = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C$$

$$I_2 = \int\frac{dx}{\operatorname{sh} x} = \int\frac{dx}{\frac{e^x - e^{-x}}{2}} = 2\int\frac{dx}{e^x - \frac{1}{e^x}} = 2\int\frac{e^x\,dx}{e^{2x} - 1} = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array}\right| =$$

$$= 2\int\frac{dt}{t^2 - 1} = 2\int\frac{dt}{(t-1)(t+1)} = \left| \begin{array}{l} \frac{1}{(t-1)(t+1)} = \frac{A}{t-1} + \frac{B}{t-1} \\ \frac{1}{(t-1)(t+1)} = \frac{1}{2}\left(\frac{1}{t-1} - \frac{1}{t+1}\right) \end{array}\right| =$$

$$= 2\cdot\frac{1}{2}\left[\int\frac{dt}{t-1} - \int\frac{dt}{t+1}\right] = \ln|t-1| - \ln|t+1| + C =$$

$$= \ln\left|\frac{t-1}{t+1}\right| + C = \ln\left|\frac{e^x - 1}{e^x + 1}\right| + C = \ln\left|\operatorname{th}\frac{x}{2}\right| + C$$

Rj. $\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$$I = \int \left(\frac{1}{\sin x} + \frac{1}{\operatorname{sh} x}\right) dx = I_1 + I_2 = \ln\left|\operatorname{tg}\frac{x}{2}\right| + \ln\left|\operatorname{th}\frac{x}{2}\right| + C$$

$$= \ln\left|\operatorname{tg}\frac{x}{2}\cdot\operatorname{th}\frac{x}{2}\right| + C$$

$$y = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\quad}\, dx$$

Neka je $\sqrt{\sin^2 x\cos x} = f(x)$. Provjerimo da li je funkcija $f(x)$ parna ili neparna.

V.i.l.

$$f(-x) = \sqrt{\sin^2(-x)\cos(-x)} = \sqrt{\sin^2 x \cos x} = f(x)$$

$$f(-x) = f(x).$$

Dakle, data funkcija je parna, pa je

$$y = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2 x \cos x}\, dx = 2\int\limits_{0}^{\frac{\pi}{2}} |\sin x|\sqrt{\cos x}\, dx$$

Kako je $\sin x$ u intervalu $\left[0, \frac{\pi}{2}\right]$ pozitivna tj.

$\sin x \geq 0$, za $x \in \left[0, \frac{\pi}{2}\right]$, to integral y možemo pisati

(oslobodit se apsolutne vrijed...

$$y = 2\int_0^{\frac{\pi}{2}} \sin x\sqrt{\cos x}\,dx \left| \begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \\ \text{granice nove:} \\ x=0,\ \cos x = 1,\ t=1 \\ x=\frac{\pi}{2},\ \cos x = 0,\ t=0 \end{array} \right| =$$

$$= 2\int_1^0 \sqrt{t}\cdot(-dt) = -2\int_1^0 t^{\frac{1}{2}}dt = -\left(-\int_0^1 t^{\frac{1}{2}}dt\right) =$$

$$= 2\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^1 = \frac{4}{3}t^{\frac{3}{2}}\Big|_0^1 = \frac{4}{3}$$

Zad 3

Izračunajmo derivaciju funkcije:

$$f(x) = \int_{-x}^{x} \frac{t^2+1}{t-1}\,dt \text{ u tački } x = \frac{1}{2}, \text{ pa:}$$

Ako je

$$f(x) = \int_{h(x)}^{g(x)} F(t)\,dt, \text{ tada je}$$

$$f'(x) = F(g(x))\cdot(g(x))' - F(h(x))\cdot(h(x))'.$$

U našem zadatku je

$$F(t) = \frac{t^2+1}{t-1}$$

$$g(x) = x; \quad g'(x) = 1; \quad F(g(x)) = \frac{x^2+1}{x-1}$$

$$h(x) = -x; \quad h'(x) = -1, \quad F(h(x)) = \frac{(-x)^2+1}{-x-1} = \frac{x^2+1}{-x-1}, \text{ pa je}$$

$$f'(x) = \frac{x^2+1}{x-1}\cdot 1 - \frac{x^2+1}{-x-1}\cdot(-1) = (x^2+1)\left(\frac{1}{x-1} + \frac{1}{-x-1}\right)$$

$$f'(x) = (x^2+1)\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$$

Kako se traži derivacija date funkcije u tački $x = \frac{1}{2}$,

to je:

$$f'\left(\frac{1}{2}\right) = \left(\left(\frac{1}{2}\right)^2+1\right)\left(\frac{1}{\frac{1}{2}-1} - \frac{1}{\frac{1}{2}+1}\right) =$$

$$= \frac{5}{4}\cdot\left(\frac{1}{-\frac{1}{2}} - \frac{1}{\frac{3}{2}}\right) = \frac{5}{4}\left(-2 - \frac{2}{3}\right) = \frac{5}{4}\left(\frac{-6-2}{3}\right)$$

tacion)

$$-16\,\frac{(4-3y^2)}{(y^2+4)^3}=0$$

$$4-3y^2=0$$

$$3y^2=4$$

$$y^2=\frac{4}{3} \Rightarrow \quad x\cdot y^2=8-4x$$

$$x\cdot\frac{4}{3}=8-4x \,/\cdot 3$$

$$4x=24-12x$$

$$16x=24$$

$$\boxed{x=\frac{3}{2}}$$

Funkcija će se x osu kada je $y=0$, tj:

$$8-4x=0$$

$$\underline{x=2}$$

Zapremina tražena je od

$$V=\int_{\frac{3}{2}}^{2} y^2\,dx=\int_{\frac{3}{2}}^{2}\frac{8-4x}{x}\,dx=$$

$$=\pi\left(\int_{\frac{3}{2}}^{2}\left(\frac{8}{x}-4\right)dx\right)=\pi\left(8\int_{\frac{3}{2}}^{2}\frac{dx}{x}-4\int_{\frac{3}{2}}^{2}dx=\right)$$

$$=\pi\left(8\ln x\Big|_{\frac{3}{2}}^{2}-4x\Big|_{\frac{3}{2}}^{2}\right)=\pi\left(8\ln\frac{2}{\frac{3}{2}}-(8-6)\right)$$

$$=\pi\left(8\ln\frac{4}{3}-2\right)=8\pi\left(\ln\frac{4}{3}-\frac{1}{4}\right)$$

$$= \frac{5}{4}\left(\frac{-8}{3}\right) =$$

$$= \frac{-10}{3}$$

$$f'\left(\frac{1}{2}\right) = -\frac{10}{3}$$

Zadatak 4

$$xy^2 = 8 - 4x$$

Nađimo prvo prevojne tačke luka $xy^2 = 8 - 4x$.
Posmatrajmo luk kao $x(y)$,

$$xy^2 = 8 - 4x$$

$$x(y^2+4) = 8 \; ; \; x = \frac{8}{y^2+4}.$$

Prevojne tačke (kada je $x^{(2)}$ (drugi izvod) jednak 0)

$$x' = \left(8\cdot(y^2+4)^{-1}\right)' = 8\cdot(-1)\cdot\frac{1}{(y^2+4)^2}\cdot 2y = -16\frac{y}{(y^2+4)^2}$$

$$x'' = -16\left(\frac{y}{(y^2+4)^2}\right)' = -16\left(\frac{(y^2+4)^2 - y\cdot 2(y^2+4)\cdot 2y}{(y^2+4)^4}\right) =$$

$$= -16\left(\frac{(y^2+4)\left[(y^2+4) - 4y^2\right]}{(y^2+4)^4}\right) =$$

$$= -16\,\frac{\left[-3y^2+4\right]}{(y^2+4)^3} = -16\,\frac{(4-3y^2)}{(y^2+4)^3}$$

Zadatak 5

$f(x) = \text{arc ctg} \frac{x+n}{x-n}$; n najmanja cifra jedinstvenog matičnog broja (1004989170017)

$n = 4$

$f(x) = \text{arc ctg} \frac{x+4}{x-4}$

a) Domen:

arc ctg α - def. za sve vrijednosti α

$\frac{x+4}{x-4}$ - def. ~~za~~ $x \neq 4$, pa je

domen tražene funkcije $x \in (-\infty, 4) \cup (4, +\infty)$

Ponašanje funkcije na rubovima područja Dom(f)

- rubovi domena

$$\lim_{x \to +\infty} \text{arcctg} \frac{x+4}{x-4} = \text{arcctg}\left(\lim_{x \to +\infty} \frac{x+4 \,/:x}{x-4 \,/:x}\right) =$$

$$= \text{arcctg}\left(\lim_{x \to +\infty} \frac{1+\frac{4}{x}}{1-\frac{4}{x}}\right) = \text{arcctg}\, 1 = \frac{\pi}{4}.$$

$$\lim_{x \to -\infty} \text{arcctg} \frac{x+4}{x-4} = \text{arcctg}\left(\lim_{x \to -\infty} \frac{x+4}{x-4}\right) = \text{arcctg}\, 1 = \frac{\pi}{4}$$

b) Presjeci grafika sa koordinatnim osama

sa y-osom

$\underline{x=0}$; $f(0) = \operatorname{arcctg} \frac{4}{-4} = \operatorname{arcctg}(-1) = ?$

$f(0) = -$

sa x-osom

$\underline{y=0}$; $0 = \operatorname{arcctg} \frac{x+4}{x-4}$

pa je

$\frac{x+4}{x-4} = \underline{ne}$

$x+4=0$; $\underline{x=-4}$

Znak funkcije:

	$-\infty$	4	$+\infty$
$f(x$	$+$	$+$	

$$\frac{1}{-\infty}$$

Asimptote

$$\lim_{x\to 4-} f(x) = \lim_{x\to 4-} \operatorname{arctg}\frac{x+4}{x-4} = \operatorname{arctg}\left(\lim_{x\to 4-}\frac{x+4}{x-4}\right) = \operatorname{arctg}(-\infty) =$$

$$\frac{8}{0^-} = -\infty$$

$$\frac{8}{+0}$$

$$\lim_{x\to 4+} f(x) = \lim_{x\to 4+} \operatorname{arctg}\frac{x+4}{x-4} = \operatorname{arctg}\left(\lim_{x\to 4+}\frac{x+4}{x-4}\right) = \operatorname{arctg}(+\infty)$$

– (za $\lim_{x\to\pm\infty} f(x)$ – smo ispitali ...)

KOSE:

$$k = \lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\operatorname{arctg}\left(\frac{x+4}{x-4}\right)}{x} = \lim_{x\to+\infty}\frac{\operatorname{arctg}\left(\frac{x+4}{x-4}\right)}{\frac{1}{x}} = \bot$$

$$\text{LOP} = \lim \frac{\frac{4}{x^2+16}}{-\frac{1}{x^2}} = \lim_{x\to\pm\infty}\frac{4x^2}{(x^2+16)} = -4$$

$$\lim (f(x) - kx) = \lim_{x\to\pm\infty}\left(\left(\operatorname{arctg}\frac{x+4}{x-4}\right) + 4x\right) = \infty$$

– ne ima kose asimpt.

c) Eventualne tačke prekida

Eventualne tačka [illegible] je u $x = 4$ i to jeste tačka prekida jer $x = 4$ [illegible] domenu funkcije

f [illegible]

$$\lim_{x \to 4-} \operatorname{arcctg} \frac{x+4}{x-4} = \operatorname{arcctg}(-\infty) = 0$$ (ispitano po asimptotama)

$$\lim_{x \to 4+} \operatorname{arcctg} \frac{x+4}{x-4} = \operatorname{arcctg}(+\infty) = 0$$ (ispitano po asimptotama)

to se radi o prekidu [illegible] vrste

Sada neka je recipročna funkcija funkcije f, data sa $g(x)$ i $g(x) = \frac{1}{f(x)} = \frac{1}{\operatorname{arcctg}\left(\frac{x+4}{x-4}\right)}$. Mora biti $x \neq 4$ i

$\operatorname{arcctg}\left(\frac{x+4}{x-4}\right) \neq 0$; tj. $\frac{x+4}{x-4} \neq k\pi$

tj. $x+4 \neq k\pi x - 4k\pi$

$x(1-k\pi) \neq -4-4k\pi$; $x \neq \frac{4k\pi+4}{k\pi-1}$.

Eventualne tačke prekida funkcije $g(x) = \frac{1}{f(x)} = \frac{1}{\operatorname{arcctg}\frac{x+4}{x-4}}$

su u $x = 4$ i $x = \frac{4k\pi+4}{k\pi-1}$

d) intervali monotonosti

(znak prvog izvoda)

$(x-4)-(x+$

$$f'(x) = \left(\operatorname{arcctg} \frac{x+4}{x-4}\right)' =$$

$$= \frac{-1}{1+\left(\frac{x+4}{x-4}\right)^2} \cdot \left(\frac{x+4}{x-4}\right)' = \frac{-\cancel{(x-4)^2}}{(x-4)^2+(x+4)^2} \cdot \frac{(x-4)-(x+4)}{\cancel{(x-4)^2}} =$$

$$= \frac{-1}{x^2-8x+16+x^2+8x+16} \cdot (-8) = \frac{8}{2x^2+32} = \frac{4}{x^2+16}$$

Kako je x^2+16 uvijek veće od nule, to je funkcija

	$-\infty$	4	$+\infty$
$f'(x)$	$\nearrow$	$\nearrow$	

$f(x) = \operatorname{arcctg} \frac{x+4}{x-4}$ uvijek rastuća.

Kako je f rastuća na čitavom domenu, to ona nema ekstrema, (nema ni prevojnih tačaka)

Ispitajmo konveksnost i konkavnost i eventualne prevojne tačke, date funkcije $f(x)$, a to ćemo ispitati tako što nađemo $f''(x)$.

$$f''(x) = \left(\frac{4}{x^2+16}\right)' = \left(4\cdot(x^2+16)^{-1}\right)' = \frac{-4}{(x^2+16)^2}\cdot 2x = \frac{-8}{(x^2+16)^2}\cdot x$$

$\Rightarrow$ $f''(x)$ ovisi samo od x (jer je $(x^2+16)^2$ uvijek > 0)

	$-\infty$ … 0	0 … 4	4 … $+\infty$
x	$-$	$0+$	$+$
$(x^2+16)^2$	$+$	$+$	$+$
$f''(x)$	$-$	$+$	$+$

Konveksna: $x \in (0,4) \cup (4,+\infty)$

Konkavna: $x \in (-\infty, 0)$

Eventualna prevojna i povratna tačka je $x = 0$.

slika