Z A D A C I - Var. B :
za drugi parcijalni ispit iz IM1, 09. 01. 2008.

Zad. 1. Izračunajte granične vrijednosti L_1 (primjenom *Taylorove formule*) i L_2 (pomoću određenog integrala) ako je $L_1 := \lim\limits_{x\to 0} \dfrac{\operatorname{tg}(\operatorname{tg} x) - \sin(\sin x)}{\operatorname{tg} x - \sin x}$, $L_2 := \lim \sum\limits_{i=1}^{n} \dfrac{n}{(i-1)^2 + n^2}$.

I. $L_1 = \frac{\pi}{4}, L_2 = \frac{\pi}{3}$. **II.** $L_1 = 1, L_2 = \frac{\pi}{3}$. **III.** $L_1 = 1, L_2 = \frac{\pi}{6}$. **IV.** $L_1 = 2, L_2 = \frac{\pi}{4}$.

Zad. 2. Za sve $a, b \in \mathbf{R} \setminus \{0\}$ odredite funkciju $F_{a,b}(x)$ tako da je

$\int (a^2 \cos^2 x + b^2 \sin^2 x)^{-1} \, dx = F_{a,b}(x) + C$, gdje je C proizvoljna realna konstanta .

....

I. $F_{a,b}(x) = \dfrac{a}{b} \operatorname{arc\,tg}\left(\dfrac{a}{b} \operatorname{tg} x\right) + \dfrac{\pi n}{|ab|}$ za $(2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}$, $F_{a,b}\left((2n\pm1)\frac{\pi}{2}\right) = \dfrac{2n\pm1}{2|ab|}$; $(n := \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor)$.

II. $F_{a,b}(x) = \dfrac{1}{ab} \operatorname{arc\,tg}\left(\dfrac{b}{a} \operatorname{tg} x\right) + \dfrac{\pi n}{|ab|}$ za $(2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}$, $F_{a,b}\left((2n\pm1)\frac{\pi}{2}\right) = \dfrac{2n\pm1}{2|ab|}$; $(n := \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor)$.

III. $F_{a,b}(x) = \dfrac{b}{a} \operatorname{arc\,tg}\left(\dfrac{b}{a} \operatorname{tg} x\right) + \dfrac{2\pi n}{|a|}$ za $(2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}$, $F_{a,b}\left((2n\pm1)\frac{\pi}{2}\right) = \dfrac{2n\pm1}{2|ab|}$; $(n := \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor)$.

IV. $F_{a,b}(x) = \dfrac{1}{ab} \operatorname{arc\,tg}\left(\dfrac{b}{a} \operatorname{tg} x\right) + \dfrac{2\pi n}{|ab|}$ za $(2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}$, $F_{a,b}\left((2n\pm1)\frac{\pi}{2}\right) = \dfrac{2n\pm1}{2|ab|}$; $(n := \left\lfloor \frac{x}{\pi} + \frac{1}{2} \right\rfloor)$.

Zad. 3. Izračunajte derivaciju prvog reda funkcije $f(x) := \int\limits_{-x}^{x} \exp(-\frac{1}{t^2}) dt$, ($x \in \mathbf{R}$), u tački $x = 10$.

I. $f'(10) = \dfrac{2}{\sqrt[10]{e}}$. II. $f'(10) = \sqrt{\dfrac{2}{e}}$. **III.** $f'(10) = \dfrac{2}{\sqrt[100]{e}}$. IV. $f'(10) = \dfrac{2}{\sqrt{e}}$.

Zad. 4. Kriva C zadana je jednačinom $xy^2 = 8 - 4x$. Izračunajte površinu P lika u ravni Oxy ograničenog lukom zadane krive C i tetivom koja spaja prevojne tačke te krive.

....

I. $P = \dfrac{4\pi}{3} - 2\sqrt{3}$. II. $P = 4\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{4}\right)$. III. $P = 8\left(\dfrac{\pi}{3} - \sqrt{3}\right)$. IV. $P = 8\left(\dfrac{\pi}{3} - \dfrac{\sqrt{3}}{2}\right)$.

....

Zad. 5. Realna funkcija f jedne realne promjenljive zadana je formulom $f(x) := \arccos \dfrac{x+n}{x-n}$,

gdje je n najmanja cifra Vašeg jedinstvenog matičnog broja koja je veća od 1.

a) Odredite prirodni domen Dom (f), a zatim ispitajte ponašanje funkcije f na rubovima područja Dom (f) i odredite njene eventualne asimptote.

b) Odredite eventualne presjeke grafika G(f) sa koordinatnim osama i ispitajte znak zadane funkcije f.

c) Odredite eventualne tačke prekida i singulariteta i klasificirajte ih za zadanu funkciju f i njenu recipročnu funkciju $\dfrac{1}{f}$.

d) Odredite intervale monotonosti i eventualne tačke lokalnog i apsolutnog ekstrema zadane funkcije f, kao i eventualne prelomne i povratne tačke njenog grafika .

e) Ispitajte konveksnost i konkavnost i odredite eventualne prevojne tačke zadane funkcije f.

f) Odredite sliku Im(f) i nacrtajte grafik zadane funkcije f.

Rješenje: ....@....................

$\cos x = 1 \cdot 0{,}5$

$\cancel{x} = \arccos x \mid = 0$

$\pm \frac{\pi}{3}$

Elektrotehnički fakultet
Univerziteta u Sarajevu

$f(x) > 0$

$\boxed{-\frac{\pi}{2}, \frac{\pi}{2}}$ $\cos > 0$

$$\arccos \frac{x+2}{x-2} > 0$$

$\arccos x = y$

$\cos y = x$

$$\frac{x+2}{x-2} > \cos 0 \,(1)$$

$$\frac{x+2}{x-2} > 1$$

$$\frac{x+2}{x-2} - 1 > 0$$

$$\frac{\cancel{x}+2-\cancel{x}+2}{x-2} > 0$$

$$\frac{4}{x-2} > 0$$

$\forall x \in Dom$ (ako gledamo $f(x)$ int $(0, \pi)$)

ali $f(x) < 0$ $\forall x$ ako posmatram interval $(-\pi, 0)$

c) tačke prekida

$\boxed{x=0}$ - tačka gomilanja domena koja mu ne pripada - singularitet

za recipročnu

$$g(x) = \frac{1}{\operatorname{arcs} \frac{x+2}{x-2}}$$

$$\operatorname{arcs} \frac{x+2}{x-2} \neq 0$$

$$\frac{x+2}{x-2} \neq 1$$

$$x+2 \neq x-2$$

$2 \neq -2$ nema ih

(g(x)) uključuje također da je i njen singularitet $x=0$

kao i domen

$D = (-\infty, 0)$

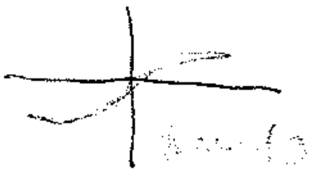
g(x) bi imala singularitet u - beskonačno

jer je za $x \to -\infty$

$$\operatorname{arcs} 1 = 0$$

$f(x) = \arccos \dfrac{\quad}{x-2}$

Elektrotehnički fakultet
Univerziteta u Sarajevu



$$f'(x) = \frac{1}{-\sqrt{1-\left(\frac{x+2}{x-2}\right)^2}} \cdot \left(\frac{x+2}{x-2}\right)'$$

$x-2-x-2$

$$= \frac{1}{-\sqrt{1-\left(\frac{x+2}{x-2}\right)^2}} \cdot \frac{(x-2-(x+2)\cdot 1)}{(x-2)^2}$$

$$= \frac{-4}{-\sqrt{1-\left(\frac{x+2}{x-2}\right)^2}\,(x-2)^2}$$ nema ekstrema

ali $$= \frac{4}{\sqrt{1-\left(\frac{x-2}{x-2}\right)^2}\,(x-2)^2} \qquad f'(x)>0$$

Kako je $f'(x)$ stalno veće od nule, to je funkcija $\arccos\frac{x+2}{x-2}$ stalno rastuća pa bi se moglo zaključiti da je apsolutni ekstrem (maximum) u krajnjoj tački njenog domena (desnoj) tj $x=0$

$$\Rightarrow \max = f(0) = \pi$$

Elektrotehnič
Univerziteta u Sar

(3.)

$$f(x) = \int_{-x}^{x} \exp\left(-\frac{1}{t^2}\right) dt \qquad x = 10$$

$$f'(x) = e^{-\frac{1}{t^2}} + e^{-\frac{1}{t^2}} = 2 \cdot e^{-\frac{1}{t^2}} = \frac{2}{e^{\frac{1}{t^2}}} = \frac{2}{t^2\sqrt{e}}$$

$$f'(10) = \frac{2}{100\sqrt{e}}$$

(4.) $xy^2 = 8 - 4x$

$$xy^2 + 4x = 8$$

$$x(y^2+4) = 8$$

$$x = \frac{8}{y^2+4}$$

$$x' = \frac{-8 \cdot 2y}{(y^2+4)^2} = \frac{-16y}{(y^2+4)^2} \qquad \frac{16 \cdot}{64}$$

$$x'' = \frac{-16(y^2+4)^{\cancel{2}} + 16y \cdot 2(\cancel{y^2+4}) \cdot 2y}{(y^2+4)^{\cancel{4}}}$$

$$= \frac{-16y^2 - 64 + 64y^2}{(y^2+4)^3}$$

$48y^2 - 64 = 0$

$8(6y^2 - 8) = 0$

$8 \cdot 2(3y^2 - 4) = 0$

$(\sqrt{3}y - 2)(\sqrt{3}y + 2) = 0$

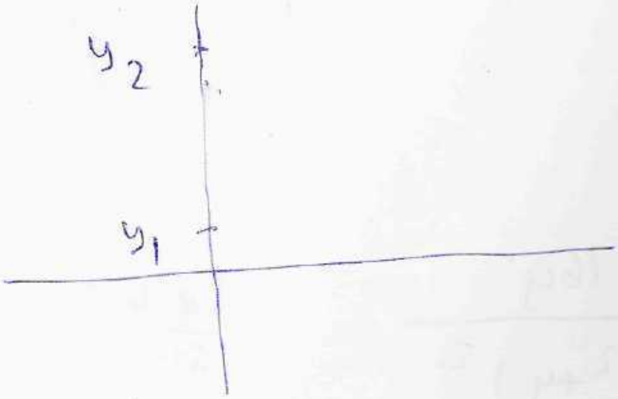
$y_1 = \frac{2}{\sqrt{3}} \quad y_2 = \frac{-2}{\sqrt{3}}$

$y_1 = \frac{2\sqrt{3}}{3} \quad y_2 = -\frac{2\sqrt{3}}{3}$

$f\left(\frac{2\sqrt{3}}{3}\right)$

$= \frac{8}{\frac{4}{3} + 4} = \frac{8}{\frac{4+12}{3}}$

$\frac{3 \cdot \cancel{8}}{\cancel{16}} = \frac{3}{2}$



$$P = \int_{y_1}^{y_2} \left(\frac{8}{y^2+4} - \frac{3}{2}\right) dx = \int_{y_1}^{y_2} \frac{8}{y^2+4} - \int_{y_1}^{y_2} \frac{3}{2}$$

Elektrotehnički fakultet
Univerziteta u Sarajevu

$$\int \frac{8}{y^2+4}\,dx = \frac{8}{4\left[\left(\frac{y}{2}\right)^2+1\right]}\,dx =$$

$$= \frac{2\cdot 8}{4}\int \frac{d\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)^2+1} = 4\,\mathrm{arctg}\,\frac{y}{2}$$

$$P = \int_{-\frac{2}{\sqrt3}}^{\frac{2}{\sqrt3}} \frac{d\left(\frac{y}{2}\right)}{\left(\frac{y}{2}\right)^2+1}\,dx \cancel{+} \int_{-\frac{2}{\sqrt3}}^{\frac{2}{\sqrt3}} \frac{3}{2}\,dx$$

$$= \cancel{4\,\mathrm{arctg}\Big|_{-\frac{2}{\sqrt3}}} \quad 4\,\mathrm{arctg}\,\frac{y}{2}\Big|_{-\frac{2}{\sqrt3}}^{2/\sqrt3} - \int_{-\frac{2}{\sqrt3}}^{\frac{2}{\sqrt3}} \frac{3}{2}\,dx$$

$$= 4\,\mathrm{arctg}\,\frac{1}{\sqrt3} + 4\,\mathrm{arctg}\,\frac{1}{\sqrt3} - \left(\frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt3} + \frac{3}{\cancel{2}}\,\frac{\cancel{2}}{\sqrt3}\right)$$

$$= 8\,\frac{\pi}{6} - \underbrace{\left(\frac{6}{\sqrt3}\right) = \left(\frac{6\sqrt3}{3}\right) =}_{2\sqrt3}$$

$$= \frac{8\pi}{6} - 2\sqrt3 = \frac{4\pi}{3} - 2\sqrt3$$

Elektrotehnički fakultet
Univerziteta u Sarajevu

(1)

b)

$$\int f(x)\,dx \sim \sum f\left(\frac{i}{n}\right)\cdot\frac{1}{n}$$

$$L_2.\quad \lim \sum_{i=1}^{n} \frac{n}{(i-1)^2+n^2} \quad \begin{matrix} : n^2 \\ : n^2 \end{matrix}$$

$$= \lim \sum_{i=1}^{n} \frac{\overbrace{\frac{1}{n}}^{dx}}{\underbrace{\left(\frac{i-1}{n}\right)}_{\xi}^{2}+1} \qquad \text{interval } (0,1)$$

$$= \int_0^1 \frac{dx}{x^2+1} = \operatorname{arctg} x \Big|_0^1$$

$$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

$$\operatorname{tg} x - \sin x = \frac{\sin x}{\cos x} - \sin x = \frac{\sin x - \sin x \cos x}{\cos x}$$

$$= \frac{\sin x(}{\quad}$$

2.

$$\int \frac{1}{a^2\cos^2 x + b^2 \sin^2 x}\,dx = \frac{:\cos^2 x}{:\cos^2 x}$$

$$\boxed{x \neq \frac{(2k-1)\pi}{2}}$$

$$= \int \frac{\frac{1}{\cos^2 x}}{a^2\left(1 + \frac{b^2}{a^2}\operatorname{tg}^2 x\right)}$$

$\frac{b}{a}\operatorname{tg} x = t$

$\frac{b}{a}\frac{1}{\cos^2 x}dx = dt$

$dx = \frac{a}{b}\cos^2 x\,dt$

$$= \frac{1}{a^2}\int \frac{\frac{a}{b}\cancel{\cos^2 x}\frac{1}{\cancel{\cos^2 x}}\,dt}{1+t^2} = \frac{1}{ab}\int\frac{dt}{1+t^2}$$

$$= \frac{1}{ab}\left[\operatorname{arctg} t + k\pi\right] = \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg} x\right) + \frac{k\pi}{|ab|}$$

$k \in \mathbb{Z}$

$$= \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg} x\right) + \frac{k\pi}{|ab|}$$

$\boxed{*1}$ $(2n-1)\frac{\pi}{2} < x < (2n+1)\frac{\pi}{2}$

6.) $f(x) = \arccos \frac{x+n}{x-n}$

$\boxed{n=2}$

$\Rightarrow f(x) = \arccos \frac{x+2}{x-2}$

a) Dom: $x \neq 2$

$-1 \leq \frac{x+2}{x-2} \leq 1$

$\frac{x+2}{x-2} - 1 \leq 0$

$\frac{x+2-x+2}{x-2} \leq 0$ $\qquad \frac{4}{x-2} \leq 0$

$\cancel{\frac{4}{x-2} \leq 0 \quad \forall x \in \mathbb{R} \setminus \{2\}}$ $\qquad x-2 \leq 0$

$\boxed{x \leq 2}$

$2 \notin$ Dom

$\cancel{\frac{x+2}{x-2} + 1 \geq 0}$ $\qquad x < 2$

Elektrotehnički fakultet
Univerziteta u Sarajevu

$$\frac{x+2}{x-2}+1\geq 0$$

$$\frac{x+2+x-2}{x-2}\geq 0$$

$$\frac{2x}{x-2}\geq 0$$

$x\in(-\infty,0]\cup(2,+\infty) \quad \wedge \quad x<2$

Dom $x\in[-\infty,0]$

$$\lim_{x\to 0} \arccos\frac{x+2}{x-2}=\lim_{x\to 0}\arccos\frac{2}{-2}=$$

$$=\arccos -1=\pi$$

$\arccos(-1)=\pi$

$$\lim_{x\to-\infty} \arccos\frac{x+2}{x-2}=1$$

$$=0$$

asimptote

nema V.A jer je

$\lim_{x\to 0} (f(x)) = \pi$ i $\lim_{x\to -\infty} = 0$

H.A. (samo kada $x \to -\infty$ zbog domene)

$\lim_{x\to -\infty} \arccos \frac{x+2}{x-2} = 1 \Rightarrow x = 0$

H.A. $y = 0$

$f(x) = 0 \Rightarrow \arccos \frac{x+n}{x-n} = 0$

$\cancel{\frac{x+n}{x-n} = (2k-1)\frac{\pi}{2}}$

$\cancel{2(x+n) = (x-n)(2k-1)\pi}$

$\cancel{2x + 2n = 2k\pi x - \pi n}$

$\cancel{2x - 2k\pi x = -\pi n - 2n}$

$\cancel{x(2 - 2k\pi) = -\pi n - 2n}$

$\cancel{x = \frac{\pi n + 2n}{2k\pi - 2} = \frac{n(\pi + 2)}{2(k\pi - 1)}}$

$\underline{za\ h=0}$

$$\cancel{\frac{x+1}{x-1} = \frac{\pi}{2}}$$

$$\cancel{x+1 = \frac{\pi}{2}(x-1)}$$

$$\cancel{x+1 = -\frac{\pi}{2} + \frac{\pi}{2}x}$$

$$\cancel{x - \frac{\pi}{2}x = -\frac{\pi}{2} - 1}$$

$$\cancel{x\left(1-\frac{\pi}{2}\right) = -\frac{\pi}{2} - 1}$$

$$\cancel{x = \frac{-\frac{\pi}{2}-1}{1-\frac{\pi}{2}} = \frac{1+\frac{\pi}{2}}{\frac{\pi}{2}-1}}$$

$$f(x) = 0$$

$$\arccos \frac{x+2}{x-2} = 0$$

$$\cancel{\frac{x+2}{x-2} = \frac{\pi}{2} + 2k\pi}$$

$$\cancel{k=0}$$

$$\cancel{x+2 = (x-2)\frac{\pi}{2}}$$

$$\frac{x+2}{x-2} = 1$$

$$\frac{x+2}{x-2} - 1 = 0$$

$$\frac{x+2-x+2}{x-2} = 0$$

$$\frac{4}{x-2} = 0$$

nema nula

za $x=0$

$$y = \arccos \frac{2}{-2} = \arccos(-1) = \pi$$