* zadaci nisi dio zvanicnog materijala, niti autor garantuje za apsolutnu tacnost; svima koji eventualno pronadju greske molim da jave autoru, na cemu se unaprijed zahvaljujem

1. Izračunati:

$$\lim_{x\to 0}\frac{(1+x)^{10}-1}{x} \quad \text{i} \quad \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$$

(Predavanja 7, 3.9.4. / str.)

$$L_1=\lim_{x\to 0}\frac{(1+x)^{10}-1}{x}=\lim_{x\to 0}\frac{e^{\ln(1+x)^{10}}-1}{x}=$$

$$=\left|\begin{array}{l}\forall A\in\mathbb{R}\\ A=e^{\ln A}\ \text{osobina}\end{array}\right|=\lim_{x\to 0}\frac{e^{10\ln(1+x)}-1}{x}\cdot\frac{10\ln(1+x)}{10\ln(1+x)}$$

$$=\lim_{x\to 0}\left(\frac{e^{10\ln(1+x)}-1}{10\ln(1+x)}\cdot\frac{10\ln(1+x)}{x}\right)=$$

$$=\left|\begin{array}{l}\text{osobina: } (*)\\ \lim\limits_{f(x)\to 0}\dfrac{e^{f(x)}-1}{f(x)}=1\end{array}\right|=10\lim_{x\to 0}\underbrace{\frac{e^{10\log(1+x)}-1}{10\ln(1+x)}}_{\xrightarrow[(*)]{}1}\cdot\underbrace{\frac{\log(1+x)}{x}}_{\xrightarrow[(**)]{}1}$$

$$L_1=10$$

Dokaz za $(**)$: $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{1}{x}\ln(1+x)=$

$$=\lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)=$$

$$= \ln e = 1$$

Treba uočiti analogiju:

$$e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

$$L_2 = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln\left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}} =$$

$$= \lim_{x\to 0} e^{\frac{1}{x}\ln\frac{a^x+b^x+c^x}{3}}$$

Prema Taylorovom razvoju $x\to 0$ tj. u okolini tačke $x_0=0$ važi:

$$e^x = 1+x+o(x), \quad x\to 0$$

$$e^{\ln a^x} = \left|\begin{matrix} x\to 0 \\ \ln a^x \to 0\end{matrix}\right| = 1+\ln a^x + o(x\ln a)$$

$$a^x = 1+x\ln a + o(x), \quad x\to 0$$

Iz toga slijedi:

$$\cancel{\frac{1}{x}\ln\frac{a^x+b^x+c^x}{3}} = \cancel{\frac{1}{x}\ln \ 1+x\ln a+o(x)+1+x\ln b}$$

$$\frac{1}{x}\ln\frac{a^x+b^x+c^x}{3}=\frac{1}{x}\ln\frac{1+x\ln a+o(x)+1+x\ln b+o(x)+1+x\ln c+o(x)}{3}$$

$$=\frac{1}{x}\ln\frac{3+x(\ln a+\ln b+\ln c)+o(x)}{3}=$$

$$=\frac{1}{x}\ln\left(1+x\ln\sqrt[3]{abc}+o(x)\right)=\frac{1}{x}\ln\left(\sqrt[3]{abc}\right)^x=$$

$$=\ln\sqrt[3]{abc}$$

Vraćamo u izraz:

$$L_2=\lim_{x\to 0}e^{\frac{1}{x}\ln\frac{a^x+b^x+c^x}{3}}=\lim_{x\to 0}e^{\ln\sqrt[3]{abc}}=\sqrt[3]{abc}$$

(2.) Dokazati da je:

$$\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3} = \frac{1}{2}$$

(Predavanja 7, 3.9.5/str.   )

(*)
$$L = \lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} =$$

$$= \lim_{x\to 0} \left(\frac{\sin x}{x} \cdot \frac{\frac{1}{\cos x} - 1}{x^2} \right) = \lim_{x\to 0} \left(\underbrace{\frac{\sin x}{x}}_{\to 1} \cdot \underbrace{\frac{1}{\cos x}}_{\to 1} \cdot \frac{1-\cos x}{x^2} \right)$$

$$= \lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{2\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2 \cdot 4} = \frac{1}{2} \lim_{x\to 0} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$$

$L = \frac{1}{2}$, š.t.d.

3. Dokazati da je

$$\lim_{x\to\pm\infty}\left(\sqrt{x^2+2x+3}-\sqrt{x^2-2x+3}\right)=\pm 2$$

(FM, 244 zj/str.)

(*)

$$L=\lim_{x\to\pm\infty}\left(\sqrt{x^2+2x+3}-\sqrt{x^2-2x+3}\right)=$$

$$=\lim_{x\to\pm\infty}\left(\left(\sqrt{x^2+2x+3}-\sqrt{x^2-2x+3}\right)\cdot\frac{\sqrt{x^2+2x+3}+\sqrt{x^2-2x+3}}{\sqrt{x^2+2x+3}+\sqrt{x^2-2x+3}}\right)$$

$$=\lim_{x\to\pm\infty}\left(\frac{x^2+2x+3-x^2+2x-3}{\sqrt{x^2+2x+3}+\sqrt{x^2-2x+3}}\right)$$

$$L_1=\lim_{x\to\infty}\left(\frac{4x}{\sqrt{x^2+2x+3}+\sqrt{x^2-2x+3}}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\right)=\left|\begin{array}{l}x>0 \text{ jer } x\to\infty\\ \sqrt{x^2}=|x|=x\end{array}\right|$$

$$=4\cdot\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{2}{x}+\frac{3}{x^2}}+\sqrt{1-\frac{2}{x}+\frac{3}{x^2}}}=4\cdot\frac{1}{2}=2$$

$$L_2=\lim_{x\to-\infty}\left(\frac{4x}{\sqrt{x^2+2x+3}+\sqrt{x^2-2x+3}}\cdot\frac{-\frac{1}{x}}{-\frac{1}{x}}\right)=\left|\begin{array}{l}x<0 \text{ jer } x\to-\infty\\ \sqrt{x^2}=|x|=-x\end{array}\right|$$

$$L_2 = 4 \lim_{x \to -\infty} \frac{-1}{\sqrt{1+\frac{2}{x}+\frac{3}{x^2}} + \sqrt{1-\frac{2}{x}+\frac{3}{x^2}}} = -4 \cdot \frac{1}{2} = -2$$

a što je i trebalo dokazati.

4. Dokazati asimptotsku relaciju

$$\sqrt[3]{1+x} - \sqrt[3]{x} \sim \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}, \quad x \to +\infty$$

(FM, 612.-2/str.)

⊛ Iskoristimo osobinu:

$$a-b = \frac{a^3-b^3}{a^2+ab+b^2}, \quad a \equiv \sqrt[3]{1+x}, \quad b \equiv \sqrt[3]{x}$$

pa slijedi:

$$\sqrt[3]{1+x} - \sqrt[3]{x} = \frac{1+x-x}{\sqrt[3]{(1+x)^2} + \sqrt[3]{(1+x)x} + \sqrt[3]{x}} =$$

$$= \frac{1}{\sqrt[3]{\left(\frac{1+x}{x}\right)^2} + \sqrt[3]{\frac{x+x^2}{x^2}} + 1} \cdot \frac{1}{\sqrt[3]{x^2}} =$$

$$\lim_{x \to +\infty} \left(\frac{1}{\underbrace{\sqrt[3]{\left(\frac{1+x}{x}\right)^2}}_{\to 1} + \underbrace{\sqrt[3]{\frac{x+x^2}{x^2}}}_{\to 1} + 1} \cdot \frac{1}{\sqrt[3]{x^2}} \right) \cong \frac{1}{3} \cdot \frac{1}{\sqrt[3]{x^2}}$$

Uporedimo dvije strane asimptotske relacije:

$$\lim_{x \to +\infty} \frac{\sqrt[3]{1+x} - \sqrt[3]{x}}{\frac{1}{3} \cdot \frac{1}{\sqrt[3]{x}}} = 3 \lim_{x \to +\infty} \frac{1}{\sqrt[3]{(1+\frac{1}{x})^2} + \sqrt[3]{1+\frac{1}{x}} + 1} =$$

$= 3 \cdot \frac{1}{3} = 1$

Kako količnik lijeve i desne strane asimptotske relacije kad $x \to +\infty$ daje rezultat 1, onda limesi brojnika i nazivnika jednako brzo rastu pa je polazna relacija zaista asimptotska.

5. Izračunati limes

$$\lim_{x\to 0} \frac{\cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx - 1}{x^2}$$

(Ušćumlić, 1978./str.)

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$$L = \lim_{x\to 0} \frac{\cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx - 1}{x^2} =$$

$$= \lim_{x\to 0} \frac{e^{\ln(\cos x \cdot \cos 2x \cdot \ldots \cdot \cos nx)} - 1}{x^2} \cdot \frac{\ln(\cos x \cdot \ldots \cdot \cos nx)}{\ln(\cos x \cdot \ldots \cdot \cos nx)}$$

$$= \lim_{x\to 0} \underbrace{\frac{e^{\ln(\cos x \cdot \ldots \cdot \cos nx)} - 1}{\ln(\cos x \cdot \ldots \cdot \cos nx)}}_{\to 1} \cdot \frac{\ln(\cos x \cdot \ldots \cdot \cos nx)}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{\ln \cos x}{x^2} + \frac{\ln \cos 2x}{x^2} + \ldots + \frac{\ln \cos nx}{x^2} \right) =$$

$$= \lim_{x\to 0} \sum_{k=1}^{n} \frac{\ln(\cos kx)}{x^2} = \left| \begin{array}{l} \text{pretpostavka} \\ \text{funkcionalni niz} \\ \text{uniformno (K) pa} \\ \text{limes i suma mogu} \\ \text{zamijeniti mjesta} \end{array} \right| =$$

$$= \sum_{k=1}^{n} \lim_{x\to 0} \frac{\ln(1 + \cos kx - 1)}{x^2} \cdot \frac{\cos kx - 1}{\cos kx - 1} =$$

$$= \sum_{k=1}^{n} \lim_{x \to 0} \underbrace{\frac{\ln(1+\cos kx - 1)}{\cos kx - 1}}_{\sim,\ \frac{\ln(1+x)}{x} \xrightarrow[x\to 0]{} 1} \cdot \frac{\cos kx - 1}{x^2} =$$

$$= \sum_{k=1}^{n} \lim_{x \to 0} \frac{\cos kx - 1}{x^2} = \left| 1 - \cos 2x = 2\sin^2 x \right| =$$

$$= \sum_{k=1}^{n} \lim_{x \to 0} \frac{-2\sin^2 \frac{kx}{2}}{x^2 \cdot \frac{k^2}{4}} \cdot \frac{k^2}{4} =$$

$$= \sum_{k=1}^{n} \lim_{x \to 0} \left(k^2 \cdot \left(-\frac{1}{2}\right) \cdot \underbrace{\left(\frac{\sin \frac{kx}{2}}{\frac{kx}{2}} \right)^2}_{\longrightarrow 1} \right) =$$

$$= -\frac{1}{2} \sum_{k=1}^{n} k^2 = \left| \begin{array}{l} 1 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \\ \text{provjereno met. indukcijom} \end{array} \right| =$$

$$= -\frac{1}{2} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{-n(n+1)(2n+1)}{12}$$

Riješiti beskonačni proizvod:

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{a_n}\right)$$

ako je opšti član a_n zadat rekurzivnom formulom:

$$\begin{cases} a_1 = 1 \\ a_{k+1} = (k+1)(1+a_k) \end{cases}$$

$$(*)\quad a_{k+1} = (k+1)(1+a_k) \Rightarrow 1+a_k = \frac{a_{k+1}}{k+1} \quad (*)$$

$$a_k = k(1+a_{k-1}) \Rightarrow \frac{a_k}{k} = 1+a_{k-1}$$

$$P_n = \prod_{k=1}^{n}\left(1+\frac{1}{a_k}\right) = \prod_{k=1}^{\infty}\frac{1+a_k}{a_k} = \left|\text{smjena } (*)\right| =$$

$$= \prod_{k=1}^{n}\frac{a_{k+1}}{(k+1)a_k} = \frac{\cancel{a_2}}{2a_1}\cdot\frac{\cancel{a_3}}{3\cancel{a_2}}\cdots\frac{a_{n+1}}{(n+1)\cancel{a_n}}$$

$$= \frac{a_{n+1}}{2\cdot 3\cdot\ldots\cdot(n+1)}\cdot\frac{1}{a_1} = \left|a_1 = 1\right| = \frac{a_{n+1}}{n!\,(n+1)} =$$

$$= \left|\text{smjena } (*)\right| = \frac{1+a_n}{n!} = \frac{1}{n!}+\frac{a_n}{n!} = \frac{1}{n!}+\frac{1}{(n-1)!}+\frac{a_{n-1}}{(n-1)!} =$$

$$= \dots = \frac{1}{n!} + \frac{1}{(n-1)!} + \dots + \frac{1}{2!} + a_1 = \left| a_1 = 1 \right|$$

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \dots + \frac{1}{2!} + \frac{1}{1!} = \sum_{k=1}^{n} \frac{1}{k!}$$

$$P = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} = \sum_{n=1}^{\infty} \frac{1}{n!} \stackrel{def}{=} e$$

Provjeriti obične i apsolutnu konvergenciju reda

$$\sum_{n=1}^{\infty} (-1)^n \left(\sqrt[n]{n}-1\right)$$

(*) Kako se radi o Leibnitzovom, alternirajućem redu za provjeru obične konvergencije primjenimo Leibnitzov kriterij:

i) $|c_{n+1}| < |c_n|$, $c_n = \sqrt[n]{n}-1$

$$\sqrt[n+1]{n+1} < \sqrt[n]{n}$$

$$(n+1)^n < n^{n+1} = n^n \cdot n$$

$$\frac{(n+1)^n}{n^n} < n$$

$$\underbrace{\left(1+\frac{1}{n}\right)^n}_{<e} < n \qquad (*)$$

$< e$ (iz definicije $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

pa je $(*)$ tačna za $n \geq 3$, $n \in \mathbb{N}$

ii) $\lim\limits_{n\to\infty} c_n = \lim\limits_{n\to\infty}\left(\sqrt[n]{n}-1\right) = \left|\lim\limits_{n\to\infty}\sqrt[n]{n} = 1\right| = 0$

pa navedeni red konvergira.

Za provjeru apsolutne konvergencije slijedi

$$\sum_{n=1}^{\infty} \left| (-1)^n (\sqrt[n]{n}-1) \right| = \sum_{n=1}^{\infty} (\sqrt[n]{n}-1)$$

Kako je $\sqrt[n]{n}-1 > \frac{1}{n}$ slijedi (provjeriti mat. indukcijom)

$$\sum_{n=1}^{\infty} (\sqrt[n]{n}-1) > \sum_{n=1}^{\infty} \frac{1}{n} \qquad (**)$$

Kako je desni red $\sum_{n=1}^{\infty} \frac{1}{n}$ Riemannov z-red, za $\alpha=1$ divergira (provjeriti logaritamskim kriterijem), to da bi važila nejednakost $(**)$ mora i lijevi red $\sum_{n=1}^{\infty} (\sqrt[n]{n}-1)$ biti divergentan, pa polazni red apsolutno divergira.

Kako polazni red obično ~~konvergira~~ konvergira, a apsolutno divergira, to kažemo da zadati red uslovno konvergira.

Naći:

i) $\ln(-2)$

ii) $\ln(1+i)$

$\circledast$ $\ln(-2)$ ne postoji u $\mathbb{R}$, ali postoji u $\mathbb{C}$ skupu.

Poslužimo se formulom:

$$\ln z = \ln|z| + i\,\mathrm{Arg}(z)$$

Kako je $\mathrm{Arg}(z) = \arg(z) + 2k\pi$, $\forall k \in \mathbb{Z}$, slijedi

$$\ln z = \ln|z| + i\arg(z) + i\cdot 2k\pi$$

pri čemu:

$$\arg(z) = \mathrm{arctg}\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}$$

i važi:

$$\mathrm{Re}\{\ln z\} = \ln|z|$$

$$\mathrm{Im}\{\ln z\} = \mathrm{Arg}(z) = \arg(z) + 2k\pi, \quad k \in \mathbb{Z}$$

Najprije, možemo pisati -2 u formi kompleksnog broja (svaki $x \in \mathbb{R}$ takođe $x \in \mathbb{C}$):

$$z = -2 \Rightarrow |z| = 2, \quad \arg(z) = \pi$$

pa je:

$$\ln(-2) = \ln 2 + i\pi + i\,2k\pi = \ln 2 + i\pi(2k+1),\ k\in\mathbb{Z}$$

Analogno važi za $\ln(1+i)$

Ispitati uslovnu i apsolutnu konvergenciju reda:

$$\sum_{n=1}^{\infty} \frac{a^n}{b^n+n}, \quad b \geq 0$$

(Uščumlić II, 131./str. 9)

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i) $|a| < 1 \Rightarrow a \in (-1,1)$

Tada važi:

$$\left|\frac{a^n}{b^n+n}\right| < |a^n|$$

Kako je red $\sum |a^n|$ konvergentan geometrijski red za $|a|<1$ to polazni red apsolutno konvergira za $\forall b \in \mathbb{R}^+ \cup \{0\}$.

ii) $0 \leq b \leq a$

ii.i) $a \neq 1$

$$\lim_{n\to\infty} \frac{a^n}{b^n+n} = \lim_{n\to\infty} \frac{1}{\left(\frac{b}{a}\right)^n + \frac{n}{a^n}} = \left| \begin{array}{l} \left(\frac{b}{a}\right)^n \xrightarrow[n\to+\infty]{} 0, \ \ b<a \text{ i } \cancel{1<b} \\ \left(\frac{b}{a}\right)^n \xrightarrow[n\to+\infty]{} \infty, \ \ b \geq 1 \end{array} \right|$$

red divergira

ii.ii) $a=1$

$$\sum_{n=1}^{\infty} \frac{1}{b^n+n}, \quad \text{maksimalno } b^n = 1, \quad b < a = 1$$

$$\frac{1}{b^n+n} \geq \frac{1}{n+1}$$

$\sum \frac{1}{n+1} \sim \sum \frac{1}{n}$ D $\Rightarrow$ i red $\sum_{n=1}^{\infty} \frac{1}{b^n+n}$ divergira

iii) $b > a$

$\frac{a^n}{b^n+n} < \frac{a^n}{b^n}$, $\sum \left(\frac{a}{b}\right)^n$ K pa je i red

$\sum \frac{a^n}{b^n+n}$ takođe K

iv) $a \leq -1$

$$\sum_{n=1}^{\infty} \frac{a^n}{b^n+n} = \sum_{n=1}^{\infty} \frac{(-1)^n |a|^n}{b^n+n}$$

iv.i) $0 \leq b \leq |a| \wedge$ ~~$a=1$~~ $a \neq -1$

nije ispunjen potreban uslov konvergencije

$\lim_{n\to\infty} \frac{|a|^n}{b^n+n} \neq 0 \Rightarrow$ D

iv.ii) ~~$a=-1$~~ $0 \leq b \leq |a| = 1$, $a=-1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{b^n+n}$ K po Leibnitzu

iv.iii) $b > |a| \Rightarrow$ apsolutno K