TRIGONOMETRIJSKE FUNKCIJE DVOSTRUKOG UGLA

Formule su:

1. $\sin 2\alpha = 2\sin\alpha\cos\alpha$
2. $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$
3. $tg 2\alpha = \dfrac{2tg\alpha}{1-tg^2\alpha}$
4. $ctg 2\alpha = \dfrac{ctg^2\alpha - 1}{2ctg\alpha}$

Primeri:

1) a) $\sin 2\alpha = \dfrac{2tg\alpha}{1+tg^2\alpha}$ Dokazati.

$\sin 2\alpha = 2\sin\alpha\cos\alpha =$(uvek možemo u imenioci dopisati 1, zar ne?)=

$\sin 2\alpha = \dfrac{2\sin\alpha\cos\alpha}{1} = \dfrac{2\sin\alpha\cos\alpha}{\sin^2\alpha+\cos^2\alpha} =$ (trik: izvučemo zajednički i gore i dole $\cos^2\alpha$)=

$$\frac{\cancel{\cos^2\alpha}\cdot\dfrac{2\sin\alpha}{\cos\alpha}}{\cancel{\cos^2\alpha}\cdot\left(\dfrac{\sin^2\alpha}{\cos^2\alpha}+1\right)} = \frac{2tg\alpha}{tg^2\alpha+1} = \frac{2tg\alpha}{1+tg^2\alpha}$$

b) $\cos 2\alpha = \dfrac{1-tg^2\alpha}{1+tg^2\alpha}$ Dokazati.

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \dfrac{\cos^2\alpha-\sin^2\alpha}{1} = \dfrac{\cos^2\alpha-\sin^2\alpha}{\sin^2\alpha+\cos^2\alpha} =$ (isti trik izvučemo $\cos^2\alpha$ i gore i dole)

$$= \frac{\cos^2\alpha\left(1-\dfrac{\sin^2\alpha}{\cos^2\alpha}\right)}{\cos^2\alpha\left(\dfrac{\sin^2\alpha}{\cos^2\alpha}+1\right)} = \frac{1-tg^2\alpha}{tg^2\alpha+1} = \frac{1-tg^2\alpha}{1+tg^2\alpha},$$ što je i trebalo dokazati.

v) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ Dokazati.

$\sin 3\alpha = \sin(2\alpha + \alpha) \rightarrow$ Iskoristimo formulu $\sin(\oplus + \square) = \sin\oplus\cos\square + \cos\oplus\sin\square$
$= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \rightarrow$ sad formule za dvostruki ugao

$= (2\sin\alpha\cos\alpha)\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\cdot\sin\alpha$
$= 2\sin\alpha\cos^2\alpha + \sin\alpha\cos^2\alpha - \sin^3\alpha$
$= 3\sin\alpha\cos^2\alpha - \sin^3\alpha$

(sad ćemo iz $\sin^2\alpha + \cos^2\alpha = 1$ izraziti $\cos^2\alpha = 1 - \sin^2\alpha$)

$= 3\sin\alpha(1 - \sin^2\alpha) - \sin^3\alpha$
$= 3\sin\alpha - 3\sin^3\alpha - \sin^3\alpha$
$= 3\sin\alpha - 4\sin^3\alpha$

g) $\cos\alpha = \dfrac{4}{5}$ Nadji vrednosti za dvostruke uglove ako je α u IV kvadrantu.

Najpre ćemo izračunati $\sin\alpha$

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\sin^2\alpha = 1 - \cos^2\alpha$$
$$\sin^2\alpha = 1 - \left(\frac{4}{5}\right)^2$$
$$\sin^2\alpha = 1 - \frac{16}{25}$$
$$\sin^2\alpha = \frac{9}{25}$$
$$\sin\alpha = \pm\sqrt{\frac{9}{25}}$$

$\sin\alpha = \pm\dfrac{3}{5}$, pošto je ugao iz IV kvadranta uzećemo da je $\sin\alpha = -\dfrac{3}{5}$

Sada je:

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\left(-\frac{3}{5}\right)\cdot\frac{4}{5}$$
$$= -\frac{24}{25}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \left(\frac{4}{5}\right)^2 - \left(-\frac{3}{5}\right)^2 = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$$
$$tg2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

2) $\sin\alpha$ =0,6 i α pripada prvom kvadrantu, nadji vrednosti za dvostruke uglove.

Sada ćemo prvo naći $\cos\alpha$

$$\sin^2\alpha + \cos^2\alpha = 1$$
$$\cos^2\alpha = 1 - \sin^2\alpha$$
$$\cos^2\alpha = 1 - (0,6)^2$$
$$\cos^2\alpha = 1 - 0,36$$
$$\cos^2\alpha = 0,64$$
$$\cos\alpha = \pm\sqrt{0,64}$$
$$\cos\alpha = \pm 0,8$$
$$\cos\alpha = +0,8$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$= 2\cdot 0,6\cdot 0,8$$
$$= 2\cdot\frac{3}{5}\cdot\frac{4}{5} = \frac{24}{25}$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$
$$= \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$
$$tg2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{\frac{24}{25}}{\frac{7}{25}}$$
$$tg2\alpha = \frac{24}{7}$$

3)Dokazati

a) $\sin 15^o \cos 15^o = \frac{1}{4}$

$\sin 15^o \cos 15^o =$ (trik je da dodamo $\frac{2}{2}$)

$= \frac{2\sin 15^0 \cos 15^0}{2} =$ (ovo u brojiocu je formula za $\sin 2\alpha = 2\sin\alpha\cos\alpha$)

$= \frac{\sin 30^o}{2} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$

b) $1 - 4\sin^2\alpha\cos^2\alpha = \cos^2 2\alpha$

$1 - 4\sin^2\alpha\cos^2\alpha =$ (pošto je formula $\sin 2\alpha = 2\sin\alpha\cos\alpha$, to je $4\sin^2\alpha\cos^2\alpha = \sin^2 2\alpha$)

pa je $1 - 4\sin^2\alpha\cos^2\alpha = 1 - \sin^2 2\alpha = \cos^2 2\alpha$

4) Dokazati

a) $2\sin^2\alpha + \cos 2\alpha = 1$

$$2\sin^2\alpha + \cos 2\alpha = 2\sin^2\alpha + \cos^2\alpha - \sin^2\alpha$$
$$= \sin^2\alpha + \cos^2\alpha = 1$$

b) $\cos^4\alpha + \sin^4\alpha = 1 - 0{,}5\sin^2\alpha$

Da bi ovo dokazali podjimo od indentiteta:

$\sin^2\alpha + \cos^2\alpha = 1 /$ Kvadriramo

$\sin^4\alpha + 2\sin^2\alpha\cos^2\alpha + \cos^4\alpha = 1$

$\sin^4\alpha + \cos^4\alpha = 1 - 2\sin^2\alpha\cos^2\alpha$ (dodamo $\frac{2}{2}$ izrazu $2\sin^2\alpha\cos^2\alpha$)

$\sin^4\alpha + \cos^4\alpha = 1 - \frac{4\sin^2\alpha\cos^2\alpha}{2}$ (ovde je $4\sin^2\alpha\cos^2\alpha = \sin^2 2\alpha$)

$\sin^4\alpha + \cos^4\alpha = 1 - \frac{1}{2}\sin^2 2\alpha$

$\sin^4\alpha + \cos^4\alpha = 1 - 0{,}5\sin^2 2\alpha$

5) Dokazati indetitet:

$$\cos 4\alpha + 4\cos 2\alpha + 3 = 8\cos^4\alpha$$

Rešenje: Poći ćemo od leve strane da dokažemo desnu.

$$\cos 4\alpha + 4\cos 2\alpha + 3 =$$
$$\cos 2\cdot(2\alpha) + 4(\cos^2\alpha - \sin^2\alpha) + 3 =$$
$$\cos^2(2\alpha) - \sin^2(2\alpha) + 4\cos^2\alpha - 4\sin^2\alpha + 3 =$$
$$(\cos^2\alpha - \sin^2\alpha)^2 - (2\sin\alpha\cos\alpha)^2 + 4\cos^2\alpha - 4\sin^2\alpha + 3 =$$
$$(\cos^2\alpha - (1-\cos^2\alpha))^2 - 4\sin^2\alpha\cos^2\alpha + 4\cos^2\alpha - 4\sin^2\alpha + 3 = [\text{zamenimo } \sin^2\alpha = 1-\cos^2\alpha]$$
$$(2\cos^2\alpha - 1)^2 - 4\cos^2\alpha(1-\cos^2\alpha) + 4\cos^2\alpha - 4(1-\cos^2\alpha) + 3 =$$
$$4\cos^4\alpha \cancel{-4\cos^2\alpha} + 1 \cancel{-4\cos^2\alpha} + 4\cos^4\alpha \cancel{+4\cos^2\alpha} - 4 \cancel{+4\cos^2\alpha} + 3 =$$
$$= 8\cos^4\alpha$$

A ovo smo trebali dokazati!!

6) Ako je $\sin\frac{x}{2} + \cos\frac{x}{2} = 1{,}4$ izračunati $\sin x$

Rešenje: Kvadriraćemo datu jednakost.

$$\sin\frac{x}{2} + \cos\frac{x}{2} = 1{,}4 \,/\, ()^2$$
$$\sin^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2} + \cos^2\frac{x}{2} = 1{,}96 \quad [\text{ovde je } 2\sin\frac{x}{2}\cos\frac{x}{2} = \sin x]$$
$$1 + \sin x = 1{,}96$$
$$\sin x = 1{,}96 - 1$$
$$\sin x = 0{,}96$$

7) Predstavi $tg3\alpha$ kao funkciju od $tg\alpha$

Rešenje:

$$tg3\alpha = tg(2\alpha + \alpha) = \frac{tg2\alpha + tg\alpha}{1 - tg2\alpha \cdot tg\alpha} =$$
$$= \frac{\frac{2tg\alpha}{1-tg^2\alpha} + tg\alpha}{1 - tg\alpha\cdot\frac{2tg\alpha}{1-tg^2\alpha}} = \frac{\frac{2tg\alpha + tg\alpha(1-tg^2\alpha)}{\cancel{1-tg^2\alpha}}}{\frac{1-tg^2\alpha+2tg^2\alpha}{\cancel{1-tg^2\alpha}}}$$
$$= \frac{2tg\alpha + tg\alpha - tg^3\alpha}{1+tg^2\alpha} = \frac{3tg\alpha - tg^3\alpha}{1+tg^2\alpha}$$

8) Dokaži indetitet:

$$\frac{1+\sin 2\alpha}{\sin\alpha+\cos\alpha}=\sqrt{2}\cos\left(\frac{\pi}{4}-\alpha\right)$$

Rešenje:

$$\frac{1+\sin 2\alpha}{\sin\alpha+\cos\alpha}=\frac{\sin^2\alpha+\cos^2\alpha+2\sin\alpha\cos\alpha}{\sin\alpha+\cos\alpha}=\frac{(\sin\alpha+\cos\alpha)^{\not 2}}{\cancel{\sin\alpha+\cos\alpha}}$$

$=\sin\alpha+\cos\alpha=$(trik: kod oba sabiraka ćemo dodati $\frac{2}{2}$ tj. $\frac{\sqrt{2}^2}{2}$)

$\frac{\sqrt{2}^2}{2}\sin\alpha+\frac{\sqrt{2}^2}{2}\cos\alpha=$ izvučemo $\sqrt{2}$ kao zajednički

$\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\alpha+\frac{\sqrt{2}}{2}\cos\alpha\right)=$ pošto je $\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ i $\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ zamenimo u izraz

$\sqrt{2}\left(\sin\frac{\pi}{4}\sin\alpha+\cos\frac{\pi}{4}\cos\alpha\right)=$ malo pretumbamo

$\sqrt{2}\left(\cos\frac{\pi}{4}\cos\alpha+\sin\alpha\sin\frac{\pi}{4}\right)=$ ovo u zagradi je formula za $\cos(\alpha-\beta)$

$\sqrt{2}\cos\left(\frac{\pi}{4}-\alpha\right)$